ESKF notes

Tuesday, January 19, 2021 9:01 AM

Quaternion definition:

$$Q = a + bi + cj + dk \in H.$$

$a, b, c, d \in R$, $i^2 = j^2 = k^2 = ijk = -1$.

In vector form:

$$q = \begin{bmatrix} q_w \\ q_v \end{bmatrix} = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix}$$

Sum:

$$p \pm q = \begin{bmatrix} p_w \\ p_v \end{bmatrix} \pm \begin{bmatrix} q_w \\ q_v \end{bmatrix} = \begin{bmatrix} p_w \pm q_w \\ p_v \pm q_v \end{bmatrix}$$

Product:

$$p \otimes q = \begin{bmatrix} p_w q_w - p_v^T q_v \\ p_w q_v + q_w p_v + p_v \times q_v \end{bmatrix}$$

Skew symmetric matrix:

$$A^T = -A, \quad [a]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

Cross product:

$$a \times b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [a]_\times b.$$

Identity:

$$q_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Conjugate:

$$q^* = \begin{bmatrix} q_w \\ -q_v \end{bmatrix}$$

$$q \otimes q^* = q^* \otimes q = q_w^2 + q_x^2 + q_y^2 + q_z^2$$

Norm:

$$\|q\| = \sqrt{q \otimes q^*} = \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2}$$

$$q \otimes q^* = \|q\|^2 \Rightarrow q \otimes \frac{q^*}{\|q\|^2} = 1 \Rightarrow q^{-1} = \frac{q^*}{\|q\|^2}$$

if $\|q\| = 1$,

$$q = \begin{bmatrix} \cos\theta \\ u\sin\theta \end{bmatrix}, \quad q^{-1} = q^* = \begin{bmatrix} \cos(-\theta) \\ u\sin(-\theta) \end{bmatrix}$$

Natural power of pure quaternions:

$$v = \begin{bmatrix} 0 \\ q_v \end{bmatrix}, \quad v = \theta u,$$

$$v^2 = -\theta^2, \quad v^3 = -u\theta^3, \quad v^4 = -\theta^4 \dots$$

Exponential map:

$$e^v = \sum_{k=0}^{\infty} \frac{v^k}{k!}$$

$$e^v = e^{\theta v} = (1 - \theta^2 + \theta^4 - \dots) + u(\theta - \theta^3 + \theta^5 - \dots) = \cos\theta + u\sin\theta.$$

Logrithm:

if $\|q\| = 1$,

$$\log q = \log \begin{bmatrix} \cos\theta \\ u\sin\theta \end{bmatrix} = \log e^{\theta u} = \begin{bmatrix} 0 \\ \theta u \end{bmatrix}$$

Exponential form

$$q^t = \exp(\log(q^t)) = \exp(t \log q)$$

if $\|q\| = 1$, $q = \begin{bmatrix} \cos\theta \\ u\sin\theta \end{bmatrix}$, $\log q = \theta u$, ∴

$$q^t = \exp(t\theta u) = \begin{bmatrix} \cos t\theta \\ u\sin t\theta \end{bmatrix}$$

Exponential map on SO3:

$$\frac{d}{dt}(R^T R) = \frac{d}{dt} I = 0$$

$$\dot{R}^T R + R^T \dot{R} = 0$$

$$R^T \dot{R} = -\dot{R}^T R = -(R^T \dot{R})$$

let $R^T \dot{R} = [w]_\times$, $w \in R^3$, $[w]_\times \in so3$,

$$(R R^T) \dot{R} = R[w]_\times$$

$$\dot{R} = R[w]_\times$$

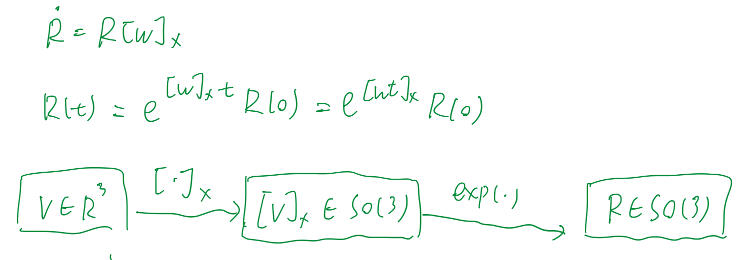
let $R(0) = I$ at $t = 0$,

$$\dot{R}(0) = [w]_\times$$

∴ $[w]_\times$ is the derivative of R at $t = 0$.

$$\dot{R} = R[w]_\times$$

$$R(t) = e^{[w]_\times t} R(0) = e^{[wt]_\times} R(0)$$



$$R = \mathrm{Exp}(v) = \exp([v]_\times)$$

Rodrigues formula:

$$R = \cos\phi I + (1 - \cos\phi) u u^T + \sin\phi [u]_\times$$

Logrithm map:

$$tr(R) = \cos\phi \, tr(I) + (1 - \cos\phi) \, tr(u u^T) + \sin\phi \, tr([u]_\times)$$

$$= 2\cos\phi + 1$$

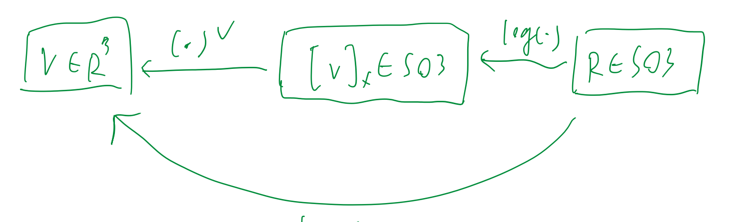
$$\therefore \phi = \arccos\left(\frac{tr(R) - 1}{2}\right)$$

$$R - R^T = 2\sin\phi [u]_\times \Rightarrow [u]_\times = \frac{R - R^T}{2\sin\phi}$$

$$\log(R) = [\phi u]_\times.$$

Define $\mathrm{Log}(\cdot): SO3 \to R^3$:

$$\mathrm{Log}(R) = \log(R)^\vee$$



Use quaternion for SO3:

$$r(v) = q \otimes v \otimes q^*, \quad \|q\|^2 = 1 \Leftrightarrow q \otimes q^* = 1.$$

$$\frac{d}{dt}(q^* \otimes q) = \dot{q}^* \otimes q + q^* \otimes \dot{q} = 0$$

$$q^* \otimes \dot{q} = -(q^* \otimes \dot{q})^*$$

$$\therefore q^* \otimes \dot{q} = \begin{bmatrix} 0 \\ \Omega \end{bmatrix} \in H_p.$$

$$q \otimes q^* \otimes \dot{q} = \dot{q} = q \otimes \Omega.$$

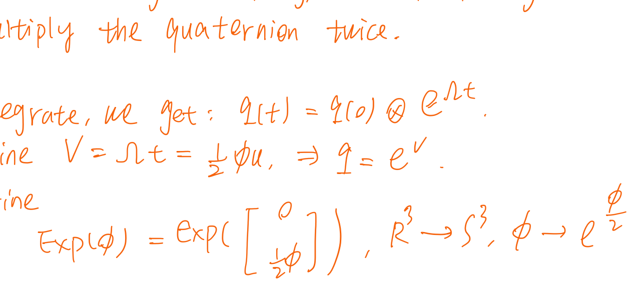
Suppose no rotation at $t = 0$, $\Rightarrow q = 1$.

$$\dot{q}(0) = \Omega \in H_p.$$

∴ pure imaginary quaternion Ω is the tangent space of S^3.
Ω is not angular velocity, but half of angular velocity, since we multiply the quaternion twice.

Integrate, we get: $q(t) = q(0) \otimes e^{\Omega t}$.
Define $V = \Omega t = \frac{1}{2}\phi u$, $\Rightarrow q = e^V$.
Define

$$\mathrm{Exp}(\phi) = \exp\left(\begin{bmatrix} 0 \\ \frac{1}{2}\phi \end{bmatrix}\right), \quad R^3 \to S^3, \quad \phi \to e^{\frac{\phi}{2}}$$



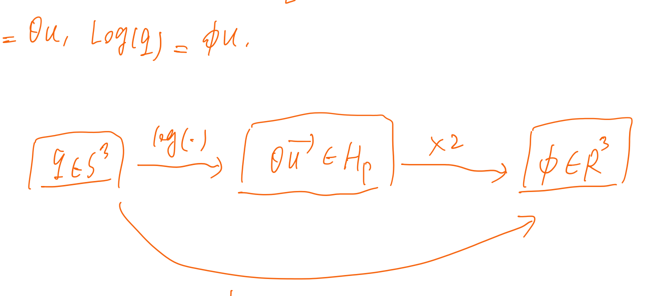
Define $w = 2\Omega$, which is angular velocity

$$\dot{q} = \frac{1}{2} q \otimes w$$

$$q = e^{\frac{w}{2}}$$

$$q = \mathrm{Exp}(\phi) = \exp\left(\frac{\phi}{2}\right) = \begin{bmatrix} \cos(\frac{\phi}{2}) \\ u\sin(\frac{\phi}{2}) \end{bmatrix}, \quad \phi = \phi u.$$

$\log q = \theta u$, $\mathrm{Log}(q) = \phi u$.



Hamilton quaternion is passive rotation, ie.

$$x_B = q \otimes x_A \otimes q^{-1} = {}_B R_A x_A.$$

is from frame A to frame B.

dis advantage:

for a vector in frame {b}, if we want to rotate it to {G}, then we need ${}_b q_G$ for passive rotation.

$$v^G = {}^b_G q \otimes v^b \otimes {}^b_G q^{-1}$$

$$= {}^b_G R v^b \rightarrow \text{but we want } R_b^G v^b!$$

∴ We need to map to transpose of rotation matrix:

$$C_H({}^b_G q) \sim R_b^G.$$

Note R_b^G is no longer frame rotation.

JPL quaternion:

difference:

$$\begin{cases} i^2 = j^2 = k^2 = -1, \\ -ij = ji = k \\ \dots \end{cases}$$

$$q = \begin{bmatrix} q_v \\ q_w \end{bmatrix}.$$

For Hamilton, it describes from {b} to {G} rotation.
For JPL, it describes from {G} to {b} frame rotation.

Note that the angle-axis for Hamilton and JPL are exactly same, but due to difference in imaginary part multiplication, one is the transpose of the other.