FEJ notes

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Notation.

True value x_l, estimate at k is $\hat{x}_l(k)$, error is $\tilde{x}_l$.

$$\hat{x}_l^{(k+n)} = {}^{(k+n)}\tilde{x}_l^{(k)} + \hat{x}_l^{(k)}.$$

$${}^{l}_{G}\hat{q}^{(k+n)} \approx (I - \lfloor {}^{(k+n)}\theta_l^{(k)} \rfloor_\times)\, {}^{l}_{G}\hat{q}^{(k)}$$

JPL quaternion is used.

IMU error dynamics:

$$\dot{\tilde{x}} = F(\hat{x})\,\tilde{x} + Gn.$$

$$\tilde{x}(t_{k+1}) = \phi(t_{k+1}, t_k)\,\tilde{x}(t_k) + \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \tau)\, G(\tau)\, n(\tau)\, d\tau$$

where $\phi(t_{k+1}, t_k) = \exp\int_{t_k}^{t_{k+1}} F(t)\, dt$

The following derivations are all at time l:

Rotation:

$$\begin{cases} {}^{I_{l+1}}_{G}R^{(l)} = {}^{I_{l+1}}_{I_l}R^{(l)}\, {}^{I_l}_{G}R^{(l)} \\ {}^{I_l}_{G}R^{(l)} = (I - \lfloor {}^{I_l}\tilde{\theta} \rfloor_\times)\, {}^{I_l}_{G}R^{(l)} \end{cases} \quad (3)$$

$$\Rightarrow \quad {}^{I_{l+1}}\tilde{\theta} = {}^{I_{l+1}}_{I_l}\tilde{\theta} + {}^{I_{l+1}}_{I_l}\hat{R}^{(l)}\, {}^{I_l}\tilde{\theta}.$$

Velocity:

$$\begin{cases} {}^{G}v_{I_{l+1}}^{(l)} = {}^{G}v^{(l)} + ({}^{I_l}_{G}R^{(l)})^T \int_{t_l}^{t_{l+1}} ({}^{I_\tau}_{I_l}R)^T a_m\, d\tau + g\Delta t \\ {}^{G}v_{I_l}^{(l)} = {}^{G}\hat{v}_{I_l}^{(l)} + {}^{G}\tilde{v}_{I_l}^{(l)} \\ {}^{I_l}_{G}R^{(l)} = (I - \lfloor {}^{I_l}\tilde{\theta} \rfloor_\times)\, {}^{I_l}_{G}\hat{R}^{(l)} \end{cases} \quad (6).$$

$$\Rightarrow \quad {}^{G}\tilde{v}_{I_{l+1}}^{(l)} = {}^{G}\tilde{v}_{I_l}^{(l)} - ({}^{I_l}_{G}\hat{R}^{(l)})^T \lfloor \hat{s}^{(l)} \rfloor_\times\, {}^{I_l}\tilde{\theta} + ({}^{I_l}_{G}\hat{R}^{(l)})^T\, \tilde{s}^{(l)}$$

Position:

$$\begin{cases} {}^{G}p_{I_{l+1}}^{(l)} = {}^{G}p_{I_l}^{(l)} + {}^{G}v_{I_l}^{(l)}\Delta t + ({}^{I_l}_{G}R^{(l)})^T \int_{t_l}^{t_{l+1}}\int_{t_l}^{s} ({}^{I_\tau}_{I_l}R)^T a_m\, d\tau\, ds + \frac{1}{2}g\Delta t^2 \\ {}^{G}p_{I_l}^{(l)} = {}^{G}\hat{p}_{I_l}^{(l)} + {}^{G}\tilde{p}_{I_l}^{(l)} \\ {}^{G}v_{I_l}^{(l)} = {}^{G}\hat{v}_{I_l}^{(l)} + {}^{G}\tilde{v}_{I_l}^{(l)} \\ {}^{I_l}_{G}R^{(l)} = (I - \lfloor {}^{I_l}\tilde{\theta} \rfloor_\times)\, {}^{I_l}_{G}\hat{R}^{(l)} \end{cases} \quad (10).$$

$$\Rightarrow \quad {}^{G}\tilde{p}_{I_{l+1}}^{(l)} = {}^{G}\tilde{p}_{I_l}^{(l)} + {}^{G}\tilde{v}_{I_l}^{(l)}\Delta t - ({}^{I_l}_{G}\hat{R}^{(l)})^T \lfloor \hat{y}^{(l)} \rfloor_\times\, {}^{I_l}\tilde{\theta} + ({}^{I_l}_{G}\hat{R}^{(l)})^T\, \tilde{y}^{(l)}$$

Propagation at time l:

$$\begin{bmatrix} {}^{I_{l+1}}_{G}\tilde{\theta}^{(l)} \\ {}^{G}\tilde{p}_{I_{l+1}}^{(l)} \\ {}^{G}\tilde{v}_{I_{l+1}}^{(l)} \end{bmatrix} = \begin{bmatrix} {}^{I_{l+1}}_{I_l}R^{(l)} & 0 & 0 \\ -({}^{I_l}_{G}R^{(l)})^T \lfloor \hat{y}_l^{(l)} \rfloor_\times & I & I\Delta t \\ -({}^{I_l}_{G}R^{(l)})^T \lfloor \hat{s}_l^{(l)} \rfloor_\times & 0 & I \end{bmatrix} \begin{bmatrix} {}^{I_l}_{G}\tilde{\theta}^{(l)} \\ {}^{G}\tilde{p}_{I_l}^{(l)} \\ {}^{G}\tilde{v}_{I_l}^{(l)} \end{bmatrix} + \begin{bmatrix} {}^{I_{l+1}}_{I_l}\tilde{\theta}^{(l)} \\ ({}^{I_l}_{G}R^{(l)})^T \lfloor \tilde{y}_l^{(l)} \rfloor \\ ({}^{I_l}_{G}R^{(l)})^T \lfloor \tilde{s}_l^{(l)} \rfloor \end{bmatrix}$$

$$\Rightarrow \quad {}^{I_{l+1}}_{G}\tilde{x}^{(l)} = \phi(x_{I_{l+1}}^{(l)}, x_{I_l}^{(l)})\, {}^{I_l}_{G}\tilde{x}^{(l)} + w^{(l)}.$$

Observation model.

l means index of camera is l.

$$\begin{cases} z_l = \pi({}^{C_l}p_{f_j}) + n_l \\ {}^{C_l}p_{f_j} = {}^{C}_{I}R\, {}^{l}_{G}R\, ({}^{G}p_{f_j} - {}^{G}p_{I_l}) + {}^{C}p_I \end{cases}$$

$$\Rightarrow \quad H_{(I_l|l)} = J_{(f_j|l)}\, {}^{C}_{I}R\, {}^{I_l}_{G}R \Big[\underbrace{\lfloor {}^{G}p_{f_j} - {}^{G}p_{I_l} \rfloor_\times ({}^{I_l}_{G}R)^T}_{\frac{\partial e}{\partial \theta}} \quad \underbrace{-I_{3\times3}}_{\frac{\partial e}{\partial p}} \quad \underbrace{0_{3\times3}}_{\frac{\partial e}{\partial v}} \Big]$$

$$H_{(f_j|l)} = J_{(f_j|l)}\, {}^{C}_{I}R\, {}^{I_l}_{G}R.$$

$$J_{(f_j|l)} = \frac{1}{Z}\begin{bmatrix} 1 & 0 & -\frac{X}{Z} \\ 0 & 1 & -\frac{Y}{Z} \end{bmatrix}.$$

Observability:

$$O = \begin{bmatrix} H_k \\ H_{k+1}\phi_k \\ \vdots \\ H_{k+m}\phi_{k+m-1}\cdots\phi_k \end{bmatrix}$$

Assume that the estimates from prediction is true and remains unchanged:

$$X_{I_l}^{l-1} = X_{I_l}^{l} = \cdots = X_{I_l}^{l+m}$$

At time l,

$$O_l = H_l\phi_{l-1}\phi_{l-2}\cdots\phi_k.$$

We can find a pattern after computing $H_l\phi_{l-1}$, $H_l\phi_{l-1}\phi_{l-2}$.

Finally we can obtain the nullspace of the observability matrix:

$$N = \begin{bmatrix} 0_3 & {}^{I_k}_{G}R\, g \\ I_3 & -\lfloor {}^{G}p_k \rfloor_\times g \\ & \vdots \\ I_3 & -\lfloor {}^{G}p_{f_N} \rfloor_\times g \end{bmatrix} \quad (23)$$

Observability matrix in reality:

At time q_{i+1}, for the l pose, its observability matrix:

1. $\phi_{l-1}(\hat{x}_{I_l}^{(l-1)}, \hat{x}_{I_{l-1}}^{(l-1)})$, $\phi_{l-2}(\hat{x}_{I_{l-1}}^{(l-2)}, \hat{x}_{I_{l-2}}^{(l-2)}) \cdots \phi_k(\hat{x}_{k+1}^{(k)}, x_{I_k}^{(k)})$, those are computed using past values, so they are not affected by update.
2. For feature P_{f_j}, $H_{l|q_i} = \dfrac{\partial e_l^{(q_{i+1})}}{\partial \hat{x}_l^{(q_i)}}$, at time q_{i+1}, variables in window are not updated yet, so we still use $\hat{x}_l^{q_i}$ during linearization.

State transition matrix:

$$\phi_{l-1}(\hat{x}_{I_l}^{(l-1)}, \hat{x}_{l-1}^{(l-1)}) = \begin{bmatrix} {}^{I_l}_{I_{l-1}}R^{(l-1)} & 0 & 0 \\ -({}^{I_{l-1}}_{G}R^{(l-1)})^T \lfloor \hat{y}_{l-1}^{(l-1)} \rfloor_\times & I & I\Delta t_{l-1} \\ -({}^{I_{l-1}}_{G}R^{(l-1)})^T \lfloor \hat{s}_{l-1}^{(l-1)} \rfloor_\times & 0 & I \end{bmatrix} \quad (25)$$

Observation matrix:

$$H_l^{(q_i)} = J_{(f_j|l)}\, {}^{C}_{I}R\, {}^{I_l}_{G}R^{(q_i)} \{ [\lfloor {}^{G}\hat{p}_{f_j} - {}^{G}\hat{p}_{I_l}^{(q_i)} \rfloor_\times ({}^{I_l}_{G}R^{(q_i)})^T \quad -I_{3\times3} \quad 0_{3\times3}] \cdots I \cdots 0 \} \quad (26)$$

observability matrix:

Still find the pattern after multiplying the first two terms.

We can see that for the same variable, estimates at two time instances are used, and there are extra disturb terms like $({}^{I_l}_{G}R^{(l-1)})^T \lfloor {}^{(q_i)}\theta_l^{(l-1)} \rfloor_\times ({}^{I_l}_{G}R^{(l-1)})$

First estimate Jacobian.

Since the variable that uses estimates from different time instances will introduce disturb terms, we need to use the estimates at the same time instance. e.g. terms like

$${}^{(q_i)}\theta_l^{(l-1)} = {}^{(l-1)}\theta_l^{(l-1)} = 0.$$

because we use same estimate at time q_i, and $l-1$, by converting $X_k^{(k+1)}$ to $X_k^{(k-1)}$.

This way, we can get an observability matrix that is close to the ideal case.

At time k, use estimate at time $k-1$, nullspace is

$$\hat{N} = \begin{bmatrix} 0_3 & {}^{I_k}_{G}\hat{R}^{(k-1)} g \\ I_3 & -\lfloor {}^{G}\hat{p}_k^{(k-1)} \rfloor_\times g \\ & \vdots \\ I_3 & -\lfloor {}^{G}\hat{p}_{f_N} \rfloor_\times g \end{bmatrix} \quad (36)$$

$\because$ (23) uses true values, those do not change with time.

(36) uses the estimate at time $k-1$, for the state at time k.

The true value and estimate at $k-1$ have different values. but they represent the same physical meaning.

$\therefore$ The first three dimensions affect the position of IMU and features, it's like shifting the system.

The last dimension affects the yaw.

In derivation below, $x = [{}^{l}_{G}q,\ {}^{G}p_l,\ {}^{G}v_l]^T$.

After prediction, we assume what we get is ideal, so it's already the true value.

At time t_0:

$$O_t = \begin{bmatrix} \lfloor {}^{G}p_{f_j} - {}^{G}p_{I_{t_0}} - {}^{G}v_{t_0}\Delta t_{t_0}^{t-1} - \frac{1}{2}g(\Delta t_{t_0}^{t-1})^2 \rfloor_\times ({}^{I_{t_0}}_{G}R)^T \\ -I \\ -I\Delta t_{t_0}^{t-1} \\ I \end{bmatrix}^T$$

nullspace is

$$N_t = \begin{bmatrix} 0_3 & {}^{I_{t_0}}_{G}R\, g \\ I_3 & -\lfloor {}^{G}p_{t_0} \rfloor_\times g \\ 0_3 & -\lfloor {}^{G}v_{t_0} \rfloor_\times g \\ I_3 & -\lfloor {}^{G}p_{f_j} \rfloor_\times g \end{bmatrix} \quad (10).$$

In the multiplication to obtain O_t, all position and velocity use the same value (ideal value), so terms get canceled out. Eg. In $\phi(t, t-1)_{11} = {}^{I_t}_{G}R({}^{I_{t-1}}_{G}R)^T$ and $\phi(t-1, t-2)_{11} = {}^{I_{t-1}}_{G}R({}^{I_{t-2}}_{G}R)^T$, when ${}^{I_{t-1}}_{G}R$ uses the same value, those can cancel out, and will not introduce extra terms.

In reality, we have a window of poses, which get updated with new observations.

So for the same velocity, position, different values at different times are used. Eg.

At time q_{i+1}, for l pose,

$$O_l^{(q_{i+1})} = M_l^{(t)} [T_l^{(q_{i+1})} + \Delta T_l^{(q_{i+1})} \quad -I_3 \quad -\Delta t_l I_3 \mid I_3]. \quad (11).$$

$T_l^{(q_{i+1})}$ is similar to ideal case.

ΔT is due to using different values at different times.

FEJ is to use the first estimate for each variable, when we do linearization

Eg. At time t: there are three options

1. Predicted value at time $t-1$: $\hat{X}_t^{(t-1)}$
2. updated value at time t: $\hat{X}_t^{(t)}$
3. updated value at later time q_i: $\hat{X}_t^{(q_i)}$

$\because$ in observability matrix, there is $\phi(\hat{x}_t, \hat{x}_{t-1})$, and for time t, we only have predicted estimate $\hat{X}_t^{(t-1)}$, so we need to use this. Eg. change observation matrix:

$$H_{f_j|t} = J_{(f_j|t)}\, {}^{C}_{I}R\, {}^{I_t}_{G}\hat{R}^{(q_i)} [\lfloor {}^{G}p_{f_j} - {}^{G}p_{I_t}^{(q_i)} \rfloor_\times ({}^{I_t}_{G}\hat{R}^{(q_i)})^T \quad -I_{3\times3} \quad 0_{3\times3} \mid I_{3\times3}]$$

$$\Rightarrow \quad H_{f_j|t} = J_{(f_j|t)}\, {}^{C}_{I}R\, {}^{I_t}_{G}\hat{R}^{(t-1)} [\lfloor {}^{G}p_{f_j} - {}^{G}p_{I_t}^{(t-1)} \rfloor_\times ({}^{I_t}_{G}\hat{R}^{(t-1)})^T \quad -I_{3\times3} \quad 0_{3\times3} \mid I_{3\times3}]$$

This way, all disturbation terms got canceled, and we get the nullspace in the same form as the ideal case:

$$N_t^{(q_i)} = \begin{bmatrix} 0_3 & {}^{I_{t_0}}_{G}\hat{R}^{(t_0)} g \\ I_3 & -\lfloor {}^{G}p_{t_0}^{(t_0)} \rfloor_\times g \\ 0_3 & -\lfloor {}^{G}v_{t_0}^{(t_0)} \rfloor_\times g \\ I_3 & -\lfloor {}^{G}p_{f_j} \rfloor_\times g \end{bmatrix} \quad (14).$$