MSCKF covariance propagation

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Consider a linear time-invariant stochastic differential equations:

$$\frac{dx(t)}{dt} = Fx(t) + Lw(t)$$

with initial condition

$x(t_0) \sim N(m_0, P_0)$.

where $w(t)$ is white noise, which has properties:

$E[w(t)] = 0$

$C_w(t,s) = E[w(t)w^T(t)] = \delta(t-s)Q$

In general, the solution of $x(t)$ would be:

$$x(t) = \exp(F(t-t_0))x_0 + \int_{t_0}^{t} \exp(F(t-s))Lw(s)\,ds$$

The expectation and covariance of $x(t)$ are

$E[x(t)] = \exp(F(t-t_0))m_0$

$$E[(x(t)-m(t))(x(t)-m(t))^T] = \exp(F(t-t_0))P_0\exp(F(t-t_0))^T + \int_{t_0}^{t}\exp(F(t-s))LQL^T\exp(F(t-s))^T ds$$

Derivation:

Mean value:

We can use the linearity of E to separate the two terms

- For the first term, we use $E[Ax] = AE[x]$ that holds for linear (deterministic) transformations A, $A = e^{(t-t_0)F}$.
- The stochastic integral of a deterministic function always has zero mean.

$$E[x(t)] = E[\underbrace{e^{(t-t_0)F}}_{\text{deterministic}}x_0] + E\left[\int_{t_0}^{t} e^{(t-s)F}L\,dw(s)\right]$$

$$= e^{(t-t_0)F}E[x_0] + 0$$

$$= e^{(t-t_0)F}m_0$$

Variance:

Itô - Isometry:

For a deterministic function $f(t)$ and with covariance $Q=1$,

$$E\left[\left(\int_{t_0}^{t} f(s)\,dw(s)\right)^2\right] = \int_{t_0}^{t}(f(s))^2 ds$$

In the multidimensional case $G(s)$ with general covariance Q,

$$E\left[\left(\int_{t_0}^{t} G(s)\,dw(s)\right)\cdot\left(\int_{t_0}^{t} G^T(s)\,dw(s)\right)\right] = \int_{t_0}^{t} G(s)QG^T(s)\,ds$$

where $G(s) = e^{(t-s)F}L$.

In short, if we have

$\dot{x} = A(t) + w$,

w is the white noise with spectral density Q, then covariance of the solution is

$$S(t) = \phi(t,0)S(0)\phi(0,t) + \int_0^t \phi(t,s)Q\phi(s,t)\,ds$$

卡尔曼滤波与组合导航原理，第3版，Page 43，

设系统方程为

$\dot{X}(t) = F(t)X(t) + G(t)w(t)$,

系统的驱动源$w(t)$为白噪声过程：

$E[w(t)] = 0$

$E[w(t)w^T(t)] = q\delta(t-\tau)$

q为$w(t)$的方差强度阵

根据线性系统理论，系统方程的离散化形式为

$$X(t_{k+1}) = \phi(t_{k+1},t_k)X(t_k) + \int_{t_k}^{t_{k+1}}\phi(t_{k+1},\tau)G(\tau)w(\tau)\,d\tau$$

一步转移阵$\phi(t_{k+1},t_k)$满足

$\dot{\phi}(t,t_k) = F(t)\phi(t,t_k)$

$\phi(t_k,t_k) = I$

求解该方程，得

$\phi(t_{k+1},t_k) = \exp\int_{t_k}^{t_{k+1}}F(t)\,dt$

当滤波周期T ($T = t_{k+1}-t_k$) 较短时，$F(t)$可近似看做常阵：

$F(t) \approx F(t_k)$, $t_k \le t < t_{k+1}$.

$\phi(t_{k+1},t_k) = e^{TF(t_k)}$ ← 一步转移阵的实时计算公式.

$\therefore \phi_{k+1,k} = I + TF_k + \frac{T^2}{2!}F_k^2 + \frac{T^3}{3!}F_k^3 + \cdots$, $F_k = F(t_k)$

连续系统的离散化处理还包括对激励白噪声过程$w(t)$的等效离散化处理

$$W_k = \int_{t_k}^{t_{k+1}}\phi(t_{k+1},\tau)G(\tau)w(\tau)\,d\tau \quad ①$$

代入 $X(t_{k+1}) = \phi(t_{k+1},t_k)X(t_k) + \int_{t_k}^{t_{k+1}}\phi(t_{k+1},\tau)G(\tau)w(\tau)\,d\tau$

$X(t_{k+1}) = \phi(t_{k+1},t_k)X(t_k) + W_k$

简写成 $X_{k+1} = \phi_{k+1,k}X_k + W_k$.

根据①式，

$$E[W_k] = \int_{t_k}^{t_{k+1}}\phi(t_{k+1},\tau)G(\tau)E[w(\tau)]\,d\tau = 0$$

$$E[W_kW_j^T] = E\left[\int_{t_k}^{t_{k+1}}\phi(t_{k+1},t)G(t)w(t)dt\cdot\int_{t_j}^{t_{j+1}}w^T(\tau)G^T(\tau)\phi^T(t_{k+1},\tau)d\tau\right]$$

$$= \int_{t_k}^{t_{k+1}}\phi(t_{k+1},t)G(t)\left[\int_{t_j}^{t_{j+1}}E[w(t)w^T(\tau)]\cdot G^T(\tau)\phi^T(t_{k+1},\tau)\,d\tau\right]dt$$

$$= \int_{t_k}^{t_{k+1}}\phi(t_{k+1},t)G(t)\left[\int_{t_j}^{t_{j+1}}q\delta(t-\tau)G^T(\tau)\phi(t_{k+1},\tau)\,d\tau\right]dt$$

$\delta(t-\tau)$为狄拉克函数，$t\in[t_k,t_{k+1})$，$\tau\in[t_j,t_{j+1})$，如果两个区间不重合，即$j\neq k$，则t和τ就不可能相等，此时积分值恒为零.

当两区间重合时，即$j=k$，

$$E[W_kW_j^T]\big|_{j=k} = \int_{t_k}^{t_{k+1}}\phi(t_{k+1},t)G(t)qG^T(t)\phi^T(t_{k+1},t)\,dt$$

所以 $E[W_kW_j]^T = Q_k\delta_{kj}$

$$\delta_{kj} = \begin{cases}1, & j=k\\ 0, & j\neq k\end{cases}$$

$$Q_k = \int_{t_k}^{t_{k+1}}\phi(t_{k+1},t)G(t)qG^T(t)\phi^T(t_{k+1},t)\,dt$$