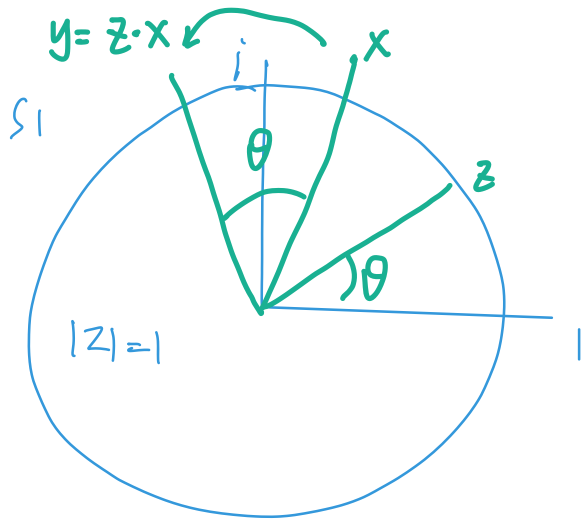
Lie theory for robotics

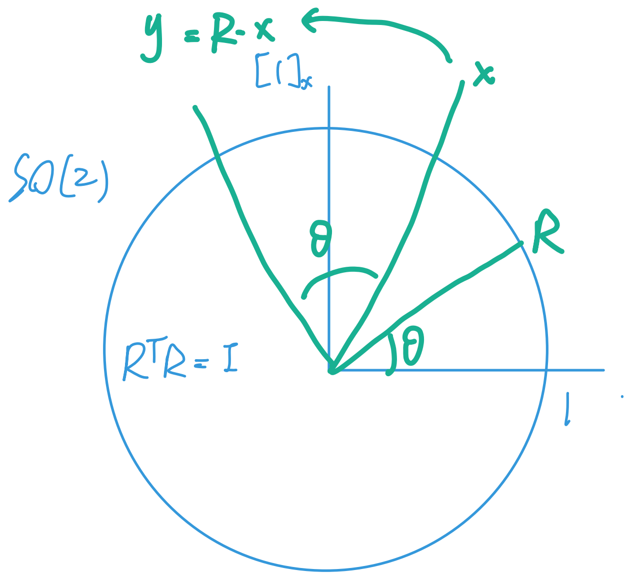
S1: The unit complex numbers



$y = z \cdot x$ rotates x, z is Lie group.

- $z^* \cdot z = 1$
- topology unit circle S1.
- $z = \cos\theta + i\sin\theta$
- inverse z^*
- composition $z_1 \cdot z_2$

SO(2)



Action: $y = R \cdot x$

Constraint: $R^T R = I$

Topology: Circle SO(2)

Elements: $R = I\cos\theta + [1]_\times \sin\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Inverse: R^T

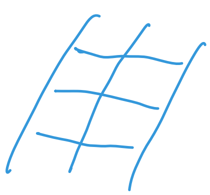
Composition: $R_1 \cdot R_2$

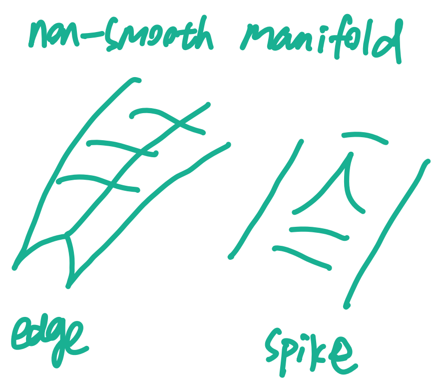
Group:

Set G of elements $\{X, Y, Z \dots\}$ with an operator '$\cdot$' such that

- composition stays in the group $X \cdot Y$ is in G.
- Identity element is in the group: $X \cdot E = E \cdot X = X$
- Inverse element is in the group: $X^{-1} \cdot X = X \cdot X^{-1} = E$
- Operation is associative: $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$

The Lie Group is also a smooth manifold.





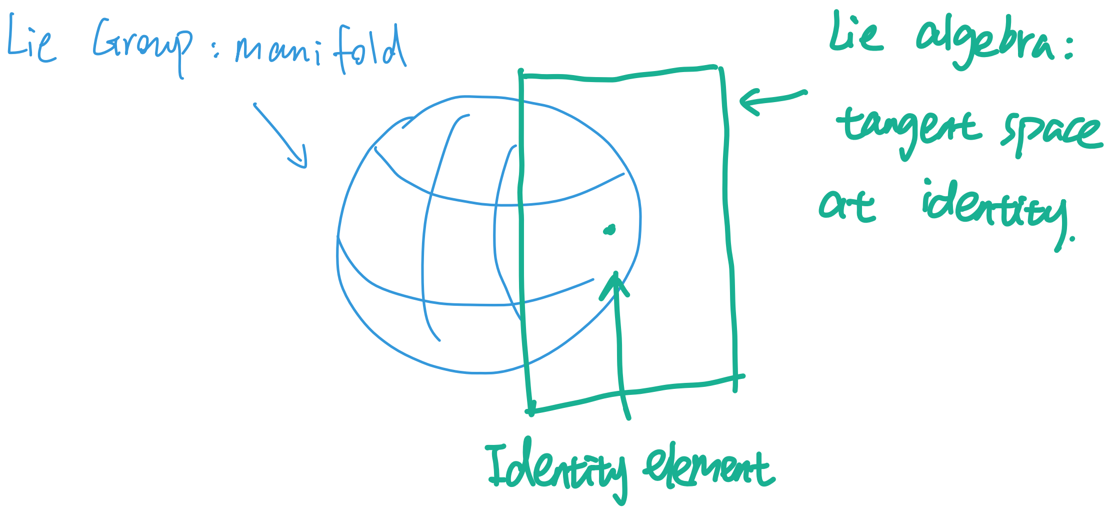
Lie Group is a smooth manifold whose elements satisfy the group axioms, also known as "Continuous Transformation groups".

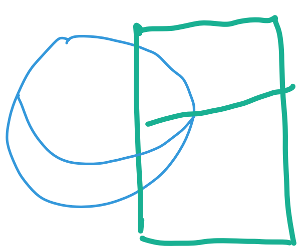
Group Action:

A group can act on another set V to transform elements:

Given X, Y in G and v in V, the action '.' is such that

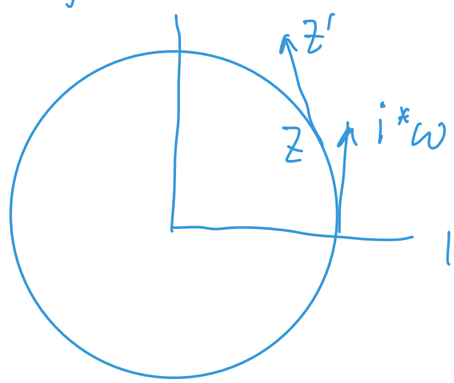
- Identity is the null action: $E \cdot v = v$
- Compatible with composition: $(X \cdot Y) \cdot v = X \cdot (Y \cdot v)$





- The operation of wraping a line on the tangent space to the arc on the sphere is the exp map. i.e. it takes vector of tangent space, and produce elements of the group.
- Inverse is the log map.

The tangent space of S^1.



Differentiate $z^* \cdot z = 1$ wrt time

$$\dot{z}^* z + \dot{z} z^* = 0$$

$$z^* \dot{z} = -(z^* \dot{z})^*$$

$$z^* \dot{z} = i\omega \in i\mathbb{R}$$

Lie Algebra: $\omega^\wedge = i \cdot \omega$ in $i\mathbb{R}$.

Cartesian: ω in $\mathbb{R}$.

Hat: $\omega^\wedge = i \cdot \omega$ Vee: $\omega = -i \cdot \omega^\vee$

The tangent space of $SO(3)$.

Differentiate $R^T \cdot R = I$, wrt time:

$$\dot{R}^T R + R^T \dot{R} = 0$$

$$R^T \dot{R} = -(R^T \dot{R})^T$$

$$R^T \dot{R} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in so(3).$$

Lie algebra when $R = I$

$$\dot{R} = \omega_\times \overset{\Delta}{=} \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in so(3)$$

Lie algebra $so(3)$:

$$w_\times = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \in so(3)$$

$$= w_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + w_y \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + w_z \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Cartesian R^3:

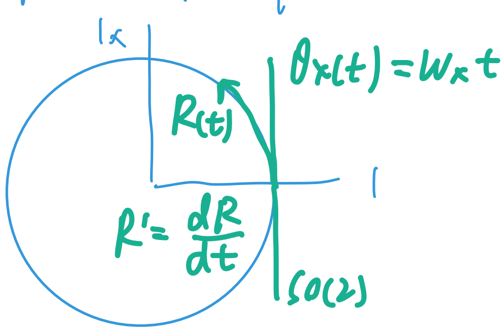
$$w = [w_x, w_y, w_z]^T \in R^3$$

$$= w_x [1,0,0]^T + w_y [0,1,0]^T + w_z [0,0,1]^T$$

Hat: $w^\wedge = w_\times$, Vee: $w = w_\times^\wedge$

The exponential map: $SO(2)$.

$SO(2)$



$$R^T \dot{R} = w_\times \Rightarrow \dot{R} = R \cdot w_\times$$

$$R(t) = R_0 \cdot \exp(w_\times t).$$

If $R_0 = R(0) = I$, and $w_\times t = \theta_\times = \theta \cdot 1_\times$,

$$R(t) = \exp(w_\times t) = \exp(\theta_\times)$$

To find closed-form expression:

$$\exp(\theta x) = I + \theta 1_x + (\theta 1_x)^2 \frac{1}{2} + (\theta 1_x)^3 \cdot \frac{1}{3!} + \ldots$$

$$= I \cdot \left(1 - \frac{\theta^3}{3!} + \ldots\right) + 1_x \cdot \left(\theta - \frac{\theta}{2!} + \ldots\right)$$

$$= I\cos\theta + 1_x \sin\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

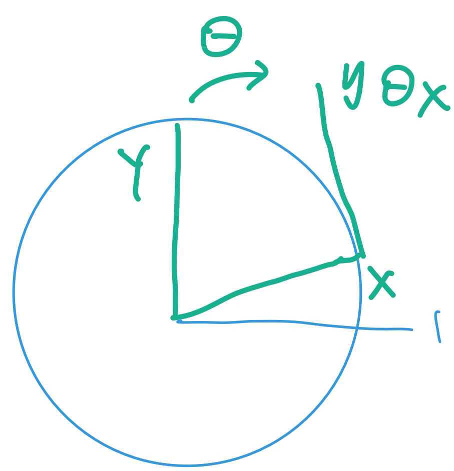
Plus and minus operators.

$$x \oplus w \triangleq x \cdot \mathrm{Exp}(w).$$

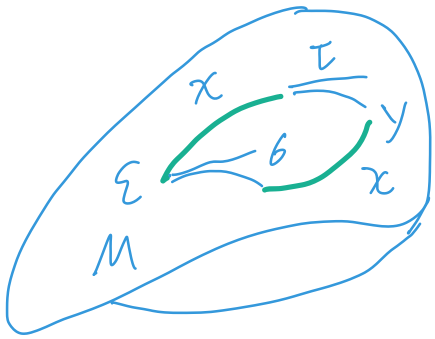
x is an element of the group.

w is an element in tangent space.

$$y \ominus x \triangleq \mathrm{Log}(x^{-1} \cdot y)$$



The adjoint matrix



$$\mathcal{Y} = \sigma \oplus \mathcal{X} = \mathcal{X} \oplus \tau$$

$$\sigma^\wedge = \mathcal{X} \cdot \tau^\wedge \cdot \mathcal{X}^{-1}$$

$$\sigma = Ad_{\mathcal{X}} \cdot \tau$$

The adjoint matrix is a linear operator that maps:

- the element of tangent space at X to
- the element of tangent space at identity E.

Jacobians on Lie Groups.

Vector space

$$J = \frac{\partial f(x)}{\partial x} \triangleq \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in R^{n \times m}$$

Lie groups

$$J = \frac{Df(\mathcal{X})}{D\mathcal{X}} = \lim_{\tau \to 0} \frac{f(\mathcal{X} \oplus \tau) \ominus f(\mathcal{X})}{\tau} \in R^{n \times m}.$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \cdots = \lim_{h\to 0} \frac{Jh}{h} \triangleq \frac{\partial Jh}{\partial h} = J$$

$$\lim_{\tau\to 0} \frac{f(x \oplus \tau) \ominus f(x)}{\tau} = \lim_{\tau\to 0} \frac{\mathrm{Log}\left[f(x)^{-1} f(x\,\mathrm{Exp}(\tau))\right]}{\tau}$$

$$= \lim_{\tau\to 0} \frac{J\tau}{\tau} \triangleq \frac{\partial J\tau}{\partial \tau} = J$$

Eg. action of $SO(3)$ on R^3:

$$f: SO(3) \times R^3 \to R^3 : (R, p) \mapsto f(R,p) = R \cdot p$$

$$\frac{Df}{DR} = \lim_{\theta\to 0} \frac{(R \oplus \theta) \cdot p - R \cdot p}{\theta}$$

operates on R^3.

$$= \lim_{\theta\to 0} \frac{(R \cdot \mathrm{Exp}(\theta)) \cdot p - R \cdot p}{\theta}$$

$$= \lim_{\theta\to 0} \frac{R \cdot (I + \theta_\times) \cdot p - R \cdot p}{\theta}$$

$$= \lim_{\theta\to 0} \frac{-R \cdot p_\times \cdot \theta}{\theta}$$

$$= -R \cdot p_\times$$

operates on R^3.

$$\frac{Df}{Dp} = \lim_{\delta p\to 0} \frac{R \cdot (p + \delta p) - R \cdot p}{\delta p}$$

$$= \lim_{\delta p\to 0} \frac{R \cdot \delta p}{\delta p} = R.$$

Differentiation rules on Lie groups.

adjoint Ad_X

right Jacobian $J_r = \frac{D\,Exp(\tau)}{D\tau}$ Inverse $\frac{DX^{-1}}{DX} = -Ad_X$

action $\frac{DX \cdot p}{DX}$, $\frac{DX \cdot p}{Dp}$ Composition: $\begin{cases} \frac{DX \cdot Y}{DX} = Ad_Y^{-1} \\ \frac{DX \cdot Y}{DY} = I \end{cases}$

Log: $\frac{D\,Log(X)}{DX} = J_r(Log(X))^{-1}$

Plus: $\frac{DX \oplus \tau}{DX} = Ad_{Exp(\tau)}^{-1}$ $\frac{DX \oplus \tau}{D\tau} = J_r(\tau)$

Chain rule:

$$\frac{DR^T p}{DR} = \frac{DR^T p}{DR^T}\frac{DR^T}{DR} = (-R^T p_\times)(-Ad_R) = R^T p_\times R$$

Perturbations on Lie Groups:

Perturbation τ over X: $\mathcal{X} = \bar{\mathcal{X}} \oplus \tau$

Covariance of X - i.e. of τ:

$$P \triangleq \mathbb{E}[\tau \cdot \tau^T]$$

$$P \triangleq \mathbb{E}[(\mathcal{X} \ominus \bar{\mathcal{X}}) \cdot (\mathcal{X} \ominus \bar{\mathcal{X}})^T]$$

Propagation:

$$y = f(x), \quad J = \frac{Dy}{Dx}, \quad P_y = J \cdot P_x \cdot J^T.$$

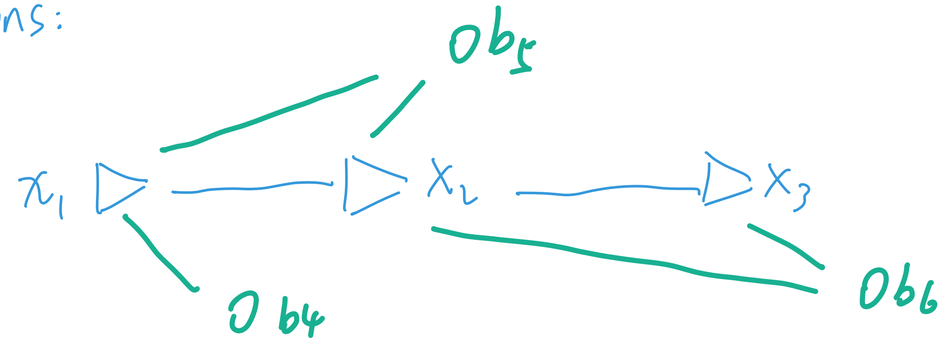
Integration on Lie Groups:

Continuous time, ω constant: $\mathcal{X}(t) = \mathcal{X}_0 \cdot \mathrm{Exp}(\omega t)$

discrete time, ω piecewise constant:

$$\mathcal{X}_4 = \mathcal{X}_0 \oplus \omega_1 dt \oplus \omega_2 dt \oplus \omega_3 dt \oplus \omega_4 dt$$

Applications:



Poses (unknown) $\mathcal{X} \sim N(\hat{\mathcal{X}}, P) \in SE(3)$

$$P = \mathbb{E}[(\mathcal{X} \ominus \bar{\mathcal{X}})(\mathcal{X} \ominus \bar{\mathcal{X}})^T]$$

covariance defined on tangent space

Beacons (known): $b_k \in \mathbb{R}^3$.

Motion model: $\mathcal{X}_i = f(\mathcal{X}_{i-1}, u_i) = \mathcal{X}_{i-1} \oplus (u_i dt + w)$

$w \sim N(0, Q)$ perturbation.

velocity is in tangent space.

Measurement model: $y_k = h(\mathcal{X}) = \mathcal{X}^{-1} \cdot b_k + v$.

$v \sim N(0, R)$ noise.

EKF prediction:

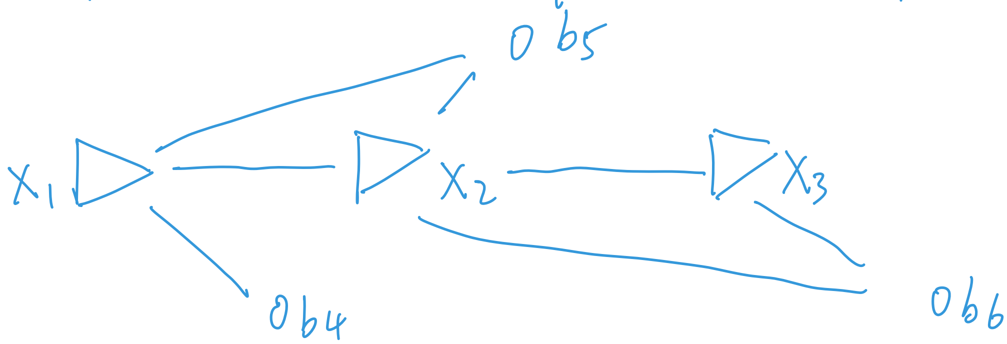
$$\hat{\mathcal{X}} \leftarrow \hat{\mathcal{X}} \oplus u_i dt \qquad F = \frac{Df}{D\mathcal{X}} \qquad G = \frac{Df}{Dw}$$

$$P \leftarrow FPF^T + GQG^T$$

EKF correction: $z_k = y_k - \hat{\mathcal{X}}^{-1} b_k$, $H = \frac{Dh}{D\mathcal{X}}$. $Z_k = HPH^T + R$

$K = PH^T Z_k^{-1}$, $\hat{\mathcal{X}} \leftarrow \hat{\mathcal{X}} \oplus K z_k$, $P \leftarrow P - K Z_k K^T$.

Graph-SLAM. Least squares on manifold.



Poses (unknown): $\mathcal{X}_i \in SE(3)$. Beacons (unknown) $b_k \in \mathbb{R}^3$.

State: Composite of Lie Groups.

$$\mathcal{X} = \langle X_1, X_2, X_3, b_4, b_5, b_6 \rangle$$

Non linear least-squares problem.

$$\mathcal{X}^* = \arg\min_{\mathcal{X}} \sum_p \| r_p(\mathcal{X}) \|^2$$

Residuals:

Prior $\quad r_1 = \Omega_1^{T/2} (\mathcal{X}_1 \ominus \mathcal{X}_1^{ref})$

Motion $\quad r_{ij} = \Omega_{ij}^{T/2} (u_j dt - (\mathcal{X}_j \ominus \mathcal{X}_i))$

Measurement $\quad r_{ik} = \Omega_{ik}^{T/2} (y_{ik} - \mathcal{X}_i^{-1} \cdot b_k)$

Update: $\delta x = -(J^T J)^{-1} J^T r$, $\quad \mathcal{X} \leftarrow \mathcal{X} \oplus \delta x$.