Continuous time IMU dynamics are

$$ {}_I\dot{R} = {}_I R({}^i\omega - b_g - n_\omega)_\times $$

$$ \dot{b}_g = n_g, \quad \dot{b}_a = n_a $$

$$ {}_I\dot{v} = {}_I R({}^i a - b_a - n_a) + g $$

$$ {}_I\dot{p} = {}_I v. $$

Deterministic nominal dynamics:

$$ {}_I\dot{\hat{R}} = {}_I\hat{R}({}^i\omega - \hat{b}_g)_\times $$

$$ {}_I\dot{\hat{v}} = {}_I\hat{R}({}^i a - \hat{b}_a) + g $$

$$ \dot{\hat{b}}_g = 0 \qquad \dot{\hat{b}}_a = 0 $$

$$ {}_I\dot{\hat{p}} = {}_I\hat{v} $$

Stochastic error dynamics:

$$ {}_I\dot{\theta} = -({}^i\omega - \hat{b}_g)_\times\, {}_I\theta - (\tilde{b}_g + n_\omega) $$

$$ {}_I\dot{\tilde{v}} = -{}_I\hat{R}({}^i a - \hat{b}_a)_\times\, {}_I\theta - {}_I\hat{R}(\tilde{b}_a + n_a) $$

$$ {}_I\dot{\tilde{p}} = {}_I\tilde{v} $$

$$ \dot{\tilde{b}}_g = n_g, \quad \dot{\tilde{b}}_a = n_a $$

Ref:

Quaternion kinematics for the error-state kalman filter.

$$\dot{\tilde{v}} = \dot{v} - \dot{\hat{v}}$$

$$= R(a-b-n) - \hat{R}(\hat{a}-\hat{b})$$

$$\approx \hat{R}(I+\theta_\times)(a-b-n) - \hat{R}(\hat{a}-\hat{b})$$

$$= \hat{R}a - \hat{R}b - \hat{R}n + \hat{R}\theta_\times a - \hat{R}\theta_\times b - \hat{R}\theta_\times n - \hat{R}\hat{a} + \hat{R}\hat{b}$$

$$= \hat{R}\theta_\times(a-b) + \hat{R}\tilde{a} - \hat{R}\tilde{b} - \hat{R}n$$

$\tilde{a} = a - \hat{a} = 0$
since $a = \hat{a}$.

$$= -\hat{R}(a-b)_\times\theta - \hat{R}(\tilde{b}+n)$$

$$\dot{R} = R(w-b-n)_\times$$

$$(\dot{\hat{R}\tilde{R}}) = R(w-b-n)_\times$$

$$\dot{\hat{R}}\tilde{R} + \hat{R}\dot{\tilde{R}} = R(w-b-n)_\times$$

$$\dot{\tilde{R}} + \hat{R}^T\dot{\hat{R}}\tilde{R} = \hat{R}^T(\hat{R}\tilde{R})(w-b-n)_\times$$

$$\dot{\tilde{R}} + \hat{R}^T\hat{R}(w-\hat{b})_\times\tilde{R} = \tilde{R}(w-b-n)_\times$$

$$\dot{\tilde{R}} = -(w-\hat{b})_\times\tilde{R} + \tilde{R}(w-b-n)_\times$$

$$\dot{\tilde{R}} = \tilde{R}(w-b-n)_\times\tilde{R}^T\tilde{R} - (w-\hat{b})_\times\tilde{R}$$

$$\dot{\tilde{R}} = [\tilde{R}(w-b-n)]_\times\tilde{R} - (w-\hat{b})_\times\tilde{R}$$

$$(Rv)_\times = R v_\times R^T$$

$$\dot{\tilde{R}} = \{[\tilde{R}(w-b-n)]_\times - (w-\hat{b})_\times\}\tilde{R}$$

$$\dot{\theta}_\times = (w-b-n+\theta_\times w-\theta_\times b-\theta_\times n-w-\hat{b})_\times (I+\theta_\times)$$

$$\dot{\theta}_\times = (-\tilde{b}-n+\theta_\times w-\theta_\times b)_\times (I+\theta_\times)$$

$$\dot{\theta}_\times = (-\tilde{b}-n+\theta_\times w-\theta_\times b)_\times$$

$$\dot{\theta} = -(w-b)_\times \theta - (\tilde{b}+n)$$

Proposition 4. The nominal dynamics can be integrated in closed-form to obtain the predicted mean $\hat{X}^p_{k+1}$:

$${}_I\hat{R}^p_{k+1} = {}_I\hat{R}_k \exp(\tau_k({}^i w_k - \hat{b}_{g,k})_\times)$$

$${}_I\hat{v}^p_{k+1} = {}_I\hat{v}_k + g\tau_k + {}_I\hat{R}_k J_L(\tau_k({}^i w_k - \hat{b}_{g,k}))({}^i a_k - \hat{b}_{a,k})\tau_k$$

$${}_I\hat{P}^p_{k+1} = {}_I\hat{P}_k + {}_I\hat{v}_k\tau_k + g\frac{\tau_k^2}{2} + {}_I\hat{R}_k H_L(\tau_k({}^i w_k - \hat{b}_{g,k}))({}^i a_k - \hat{b}_{a,k})\tau_k^2$$

$$\hat{b}^p_{g,k+1} = \hat{b}_{g,k}$$

$$\hat{b}^p_{a,k+1} = \hat{b}_{a,k}$$

$${}_I\hat{T}^p_k = \begin{bmatrix} {}_I\hat{R}_k & {}_I\hat{P}_k \\ 0^T & 1 \end{bmatrix}$$

$${}_I\hat{T}^p_{k-1} = {}_I\hat{T}_{k-1}, \quad \dots \quad {}_I\hat{T}^p_{k-w+1} = {}_I\hat{T}_{k-w+1}$$

$J_L(w) \triangleq I_3 + \frac{w_\times}{2!} + \frac{w_\times^2}{3!} + \cdots$ is left jacobian of $SO(3)$.

$H_L(w) \triangleq \frac{I_3}{2!} + \frac{w_\times}{3!} + \cdots$

Both $J_L(w)$, $H_L(w)$ has closed-form (Rodrigues) expressions:

$$J_L(w) = I_3 + \frac{1-\cos\|w\|}{\|w\|^2} w_\times + \frac{\|w\| - \sin\|w\|}{\|w\|^3} w_\times^2$$

$$H_L(w) = \frac{1}{2} I_3 + \frac{\|w\| - \sin\|w\|}{\|w\|^3} w_\times + \frac{2(\cos\|w\| - 1) + \|w\|^2}{2\|w\|^4} w_\times^2$$

Proof:

Lemma 3:

$w \in R^3$, $J_L(w) \triangleq \sum_{n=0}^{\infty} \frac{w_\times^n}{(n+1)!}$, $H_L(w) \triangleq \sum_{n=0}^{\infty} \frac{w_\times^n}{(n+2)!}$. J_L, H_L have closed form expression as above.

Proof:

$$w_\times^{2n+1} = (-1)^n \|w\|^{2n} w_\times$$

$$\therefore J_L(w) = I_3 + \sum_{n+1}^{\infty} \frac{w_\times^n}{(n+1)!}$$

$$= I_3 + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \|w\|^{2n}}{(2n+2)!}\right) w_\times + \left(\sum_{n=0}^{\infty} \frac{(-1)^n \|w\|^{2n}}{(2n+3)!}\right) w_\times^2$$

$$= I_3 + \left(\frac{1-\cos\|w\|}{\|w\|^2}\right) w_\times + \left(\frac{\|w\| - \sin\|w\|}{\|w\|^3}\right) w_\times^2$$

More details:

let $w = \phi a$, $\phi = \|w\|$, $a = \frac{w}{\phi}$.

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\phi a_\times)^n = I_3 + \frac{\phi a_\times}{2!} + \frac{\phi^2 a_\times^2}{3!} + \frac{\phi^3 a_\times^3}{4} + \cdots$$

$$= I_3 + \frac{\phi}{2!} a_\times + \frac{\phi^2}{3!} a_\times^2 + \frac{\phi^3}{4!}(-a_\times) + \frac{\phi^4}{5!}(-a_\times a_\times) + \frac{\phi^5}{6!} a_\times + \cdots$$

$$= I_3 + \left(\frac{\phi}{2!} - \frac{\phi^3}{4!} + \frac{\phi^5}{6!} - \cdots\right) a_\times + \left(\frac{\phi^2}{3!} - \frac{\phi^4}{5!} + \frac{\phi^6}{7!} - \cdots\right) a_\times a_\times$$

where we use $a_\times a_\times a_\times = -a_\times$.

$\because$ $$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \frac{\phi^8}{8!} - \cdots$$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \frac{\phi^9}{9!} - \cdots$$

$\therefore$ $$\frac{\phi}{2!} - \frac{\phi^3}{4!} + \frac{\phi^5}{6!} - \cdots = \frac{1}{\phi}\left(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \frac{\phi^6}{6!} - \cdots\right)$$

$$= \frac{1}{\phi}(1 - \cos\phi)$$

$$\frac{\phi^2}{3!} - \frac{\phi^4}{5!} + \frac{\phi^6}{7!} - \cdots = \frac{1}{\phi}\left(\frac{\phi^3}{3!} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - \cdots\right)$$

$$= \frac{1}{\phi}(\phi - \sin\phi)$$

$$a_\times = \frac{w_\times}{\phi}$$

$\therefore$ $$\sum_{n=0}^{\infty} \frac{w_\times^n}{(n+1)!} = I_3 + \frac{1-\cos\|w\|}{\|w\|^2} w_\times + \frac{\|w\| - \sin\|w\|}{\|w\|^3} w_\times^2$$

Lemma 4.

For $w \in R^3$, $t \in R$, the matrix $\exp(t w_\times)$ satisfies

$$\int_0^t \int_0^s \exp(r w_\times)\, dr\, ds = \int_0^t s\, J_L(sw)\, ds$$

$$= t^2 H_L(t\omega)$$

Proof: The result follows by integrating the terms of Taylor series of

$$\exp(t\omega_\times) = \sum_{n=0}^{\infty} \frac{t^n \omega_\times^n}{n!}$$

To predict the mean, we compute the solutions to the nominal dynamics:

$$\omega = {}^{i}\omega_k - \hat{b}_{g,k} \quad , t \in [0, \tau_k)$$

$${}_I\dot{\hat{R}} = {}_I\hat{R}\omega_\times \, , \quad {}_I\hat{R}(0) = {}_I\hat{R}_k .$$

Solution: ${}_I\hat{R}(t) = {}_I\hat{R}_k \exp(t\omega_\times)$

$$\therefore {}_I\hat{R}^p_{k+1} = {}_I\hat{R}(\tau_k) = {}_I\hat{R}_k \exp(\tau_k({}^{i}\omega_k - \hat{b}_{g,k})_\times).$$

$$a = {}^{i}a - \hat{b}_a, \quad t \in [0, \tau_k)$$

$${}_I\hat{v}(0) = {}_I\hat{v}_k . \quad {}_I\dot{\hat{v}} = {}_I\hat{R}({}^{i}a - \hat{b}_a) + g .$$

Solution: ${}_I\hat{v}(t) = {}_I\hat{v}(0) + \int_0^t ({}_I\hat{R}(s) a + g) \, ds$

$$= {}_I\hat{v}_k + t \, {}_I\hat{R}_k J_L(t\omega) a + tg .$$

Lemma 4

$$\therefore {}_I\hat{v}^p_{k+1} = {}_I\hat{v}(\tau_k).$$

More details:

$$
\begin{aligned}
{}_I\hat{v}^{p}_{t+1} &= {}_I\hat{v}_t + \int_t^{t+\tau} {}_I\dot{\hat{v}}_t \, ds \\
&= {}_I\hat{v}_t + \int_t^{t+\tau} {}_I\hat{R}_t ({}^i a - \hat{b}_a) \, ds + \int_t^{t+\tau} g \, ds \\
&= {}_I\hat{v}_t + g\tau + \int_t^{t+\tau} {}_I\hat{R}_t ({}^i a - \hat{b}_a) \, ds \\
&= {}_I\hat{v}_t + g\tau + \int_t^{t+\tau} {}_I\hat{R}_t \, ds \, ({}^i a - \hat{b}_a)
\end{aligned}
$$

$$
\begin{aligned}
\int_t^{t+\tau} {}_I\hat{R}_t \, ds &= {}_I\hat{R}_t \int_t^{t+\tau} \exp(s({}^i w_t - \hat{b}_{g,t})_\times) \, ds \\
&= R \int_t^{t+\tau} \sum_{n=0}^{\infty} \frac{1}{n!} (s(w-b)_\times)^n \, ds \\
&= R \sum_{n=0}^{\infty} \frac{1}{n!} \int_t^{t+\tau} (s(w-b)_\times)^n \, ds \\
&= R \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \tau^{n+1} ((w-b)_\times)^n \\
&= R\tau \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\tau(w-b)_\times)^n
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\tau({}^i w_t - \hat{b}_{g,t})_\times)^n = J_l(\tau({}^i w_t - \hat{b}_{g,t}))
$$

By Lemma 4, for $t \in [0, T_k)$, ${}_I\hat{p}(0) = {}_I\hat{p}_k$, ${}_I\dot{\hat{p}} = {}_I\hat{v}$,

Solution is:

$$
{}_I\hat{p}(t) = {}_I\hat{p}(0) + \int_0^t {}_I\hat{v}(s) \, ds
$$

$$= {}_I\hat{P}_k + t\,{}_I\hat{v}_k + t^2\,{}_I\hat{R}_k H_L(t\omega)\, a + \frac{t^2}{2} g.$$

$$\therefore {}_I\hat{P}^P_{k+1} = {}_I\hat{P}(T_k).$$

More details:

$${}_I\hat{P}^P_{t+1} = {}_I\hat{P}_t + \int_t^{t+\tau} {}_I\hat{\dot{P}}_t \, ds$$

$$= P + \int_t^{t+\tau} v + gs + RJ_L(s(\omega-b))(a-b)\, s\, ds$$

We need to show

$$\int_t^{t+\tau} J_L(s(\omega-b))\, s\, ds = \tau^2 H_L(\tau(\omega-b)) \quad (\#)$$

Let $\omega' = ({}^i\omega_t - \hat{b}_{g,t})$.

$$(\#) = \int_t^{t+\tau} J_L(s(\omega-b))\, s\, ds$$

$$= \int_t^{t+\tau} \left(I_3 + \frac{s\omega'^{\wedge}_x}{2!} + \frac{(s\omega'^{2}_x)}{3!} + \cdots \right) s\, ds$$

$$= \int_t^{t+\tau} \left(sI_3 + \frac{s^2\omega'_x}{2!} + \frac{s^3\omega'^2_x}{3!} + \cdots \right) ds$$

$$= \frac{\tau^2}{2} I_3 + \frac{\tau^3\omega'_x}{3\times 2!} + \frac{\tau^4\omega^2_x}{4\times 3!} + \cdots$$

$$= \tau^2 \left[\frac{I_3}{2} + \frac{\tau\omega'_x}{3!} + \frac{\tau^2\omega'^2_x}{4!} + \cdots \right)$$

$$= \tau^2 H_L(\tau(\omega-b))$$

For the closed form of H_L. let $w = \tau(\hat{\omega}_t - \hat{b}_{g,t})$, $w = \phi a$.

$$\frac{I_3}{2!} + \frac{w_\times}{3!} + \frac{w_\times^2}{4!} + \cdots$$

$$= \frac{I_3}{2} + \frac{\phi a_\times}{3!} + \frac{(\phi a_\times)^2}{4!} + \frac{(\phi a_\times)^3}{5!} + \frac{(\phi a_\times)^4}{6!} + \cdots$$

$$= \frac{I_3}{2} + \frac{\phi a_\times}{3!} + \frac{(\phi a_\times)^2}{4!} - \frac{\phi^3}{5!} a_\times - \frac{\phi^4}{6!} a_\times^2 + \cdots$$

$$= \frac{I_3}{2} + \left(\frac{\phi}{3!} - \frac{\phi^3}{5!} + \cdots\right) a_\times + \left(\frac{\phi^2}{4!} - \frac{\phi^4}{6!} + \cdots\right) a_\times^2$$

$$\because \frac{\phi}{3!} - \frac{\phi^3}{5!} + \frac{\phi^5}{7!} - \cdots = \frac{1}{\phi^2}\left(\frac{\phi^3}{3!} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - \cdots\right)$$

$$= \frac{1}{\phi^2}(\phi - \sin\phi)$$

$$\therefore \left(\frac{\phi}{3!} - \frac{\phi^3}{5!} + \frac{\phi^5}{7!} - \cdots\right) a_\times = \frac{1}{\phi^3}(\phi - \sin\phi) w_\times$$

$$\because \frac{\phi^2}{4!} - \frac{\phi^4}{6!} + \frac{\phi^6}{8!} - \cdots = \frac{1}{\phi^2}\left(\frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \frac{\phi^8}{8!} - \cdots\right)$$

$$= \frac{1}{\phi^2}\left(-\frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \frac{\phi^8}{8!} - \cdots + \frac{\phi^2}{2!}\right)$$

$$= \frac{1}{2\phi^2}\left(2(\cos\phi - 1) + \phi^2\right)$$

$$\therefore \left(\frac{\phi^2}{4!} - \frac{\phi^4}{6!} + \frac{\phi^6}{8!} - \cdots\right) a_\times^2 = \frac{1}{2\phi^4}\left(2(\cos\phi - 1) + \phi^2\right) w_\times^2$$

$$\Rightarrow H_L(w) = \frac{1}{2} I_3 + \frac{\|w\| - \sin\|w\|}{\|w\|^3} w_\times + \frac{2(\cos\|w\| - 1) + \|w\|^2}{2\|w\|^4} w_\times^2$$

To compute the predicted covariance Σ^p_{k+1}, we need to integrate the error dynamics. The IMU error state ${}_I\tilde{x}(t)$ satisfies a linear time variant (LTV) stochastic differential equation (SDE) for $t \in [0, \tau_k)$:

$${}_I\dot{\tilde{x}} = F(x)\,{}_I\tilde{x} + {}_I n, \quad {}_I\tilde{x}(0) \sim N(0, {}_I\Sigma_k)$$

${}_I n$ is white Gaussian noise with PSD

$$Q = \begin{bmatrix} \sigma_w^2 I_3 & & & & \\ & \sigma_a^2 I_3 & & & \\ & & 0 & & \\ & & & \sigma_g^2 I_3 & \\ & & & & \sigma_a^2 I_3 \end{bmatrix}$$

and $F(t)$ is

$$\begin{bmatrix} -(\omega - b)_\times & 0 & 0 & -I & 0 \\ -R(t)\,(a-b)_\times & 0 & 0 & 0 & -R(t) \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\quad {}_I\hat{R}(t) = {}_I\hat{R}_k \exp\left(t\,({}^i w_k - \hat{b}_{g,k})_\times\right)$

$\because \mathbb{E}[{}_I\tilde{x}(t)]$ is 0 over $[0, \tau_k)$, covariance of ${}_I\tilde{x}(\tau_k)$ is

${}_I\Sigma_{k+1}^{P} = \mathbb{E}\left[{}_I\tilde{x}(\tau_k)\,{}_I\tilde{x}(\tau_k)^T\right]$

$$= \Phi(\tau_k, 0)\, {}_I\Sigma_k\, \Phi(\tau_k, 0)^T + \int_0^{\tau_k} \Phi(\tau_k, s)\, Q\, \Phi(\tau_k, s)^T ds.$$

where $\Phi(t,s)$ is the transition matrix of ${}_I\dot{\tilde{x}} = F(t)\,{}_I\tilde{x} + {}_I n$.

Proposition 5. The LTV SDE has a closed form transition matrix:

$$\Phi(t,0) = \begin{bmatrix} \exp(-t\omega_\times) & 0 & 0 & -tJ_L(-t\omega) & 0 \\ \Phi_{v\theta}(t) & I_3 & 0 & \Phi_{v\omega}(t) & \Phi_{va}(t) \\ \Phi_{p\theta}(t) & tI_3 & I_3 & \Phi_{p\omega}(t) & \Phi_{pa}(t) \\ 0 & 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & 0 & I_3 \end{bmatrix}$$

where $\omega = {}^i\omega_k - \hat{b}_{g,k}$, $a = {}^i a_k - \hat{b}_{a,k}$

Proof: The transition matrix $\Phi(t,0)$ can be determined by computing the solution

$${}_I\tilde{x}(t) = \Phi(t,0)\,{}_I\tilde{x}(0)$$

to the homogeneous system

$${}_I\dot{\tilde{x}} = F(t)\,{}_I\tilde{x}$$

for initial condition.

$${}_I\tilde{X}(0) = ({}_I\theta(0), {}_I\tilde{V}(0), {}_I\tilde{p}(0), \tilde{b}_g(0), \tilde{b}_a(0))$$

The bias terms remain constant in the homogeneous system:

$$\tilde{b}_g(t) = \tilde{b}_g(0) = [0 \ \ 0 \ \ 0 \ \ I_3 \ \ 0]\ {}_I\tilde{X}(0)$$

$$\tilde{b}_a(t) = \tilde{b}_a(0) = [0 \ \ 0 \ \ 0 \ \ 0 \ \ I_3]\ {}_I\tilde{X}(0).$$

Next, consider ${}_I\dot{\theta}(t) = -\omega_\times\, {}_I\theta(t) - \tilde{b}_g(t)$ with $\omega = {}^i\omega_k - \hat{b}_{g,k}$ which is a LTI system in ${}_I\theta(t)$. Using $\tilde{b}_g(t) = \tilde{b}_g(0)$ + Lema 4. the system has solution:

$$\begin{aligned}{}_I\theta(t) &= \exp(-t\omega_\times)\, {}_I\theta(0) - \int_0^t \exp(-(t-s)\omega_\times)\, ds\, \tilde{b}_g(0) \\ &= [\exp(-t\omega_\times) \ \ 0 \ \ 0 \ \ -tJ_L(-t\omega) \ \ 0]\ {}_I\tilde{X}(0)\end{aligned}$$

Next, consider ${}_I\dot{\tilde{V}}(t) = -{}_I\hat{R}(t)\, a_\times\, {}_I\theta(t) - {}_I\hat{R}(t)\tilde{b}_a(t)$ with $a = {}^i a_k - \hat{b}_{a,k}$, which is an LTI system in ${}_I\tilde{V}(t)$. Using $\tilde{b}_a(t) = \tilde{b}_a(0)$, the LTI system has solution:

$${}_I\tilde{V}(t) = {}_I\tilde{V}(0) - \int_0^t {}_I\hat{R}(s)\, a_\times\, {}_I\theta(s)\, ds - \int_0^t {}_I\hat{R}(s)\, ds\, \tilde{b}_a$$

$$= \left[\Phi_{v\theta}(t) \quad I_3 \quad 0 \quad \Phi_{v\omega}(t) \quad \Phi_{va}(t) \right] {}_I\tilde{X}(0)$$

$$\boxed{\begin{aligned} \Phi_{v\theta}(t) &= -\int_0^t \hat{R}(s) a_\times \exp(-s\omega_\times) ds \\ \Phi_{v\omega}(t) &= \int_0^t s \hat{R}(s) a_\times J_L(-s\omega) ds \\ \Phi_{va}(t) &= -\int_0^t \hat{R}(s) ds = -t\hat{R} J_L(t\omega). \end{aligned}}$$

$\Phi_{v\theta}(t)$ Proof:

We use

1. the solution to ${}_I\dot{\hat{R}} = {}_I\hat{R}\omega_\times$, ${}_I\hat{R}(0) = {}_I\hat{R}_K$.

2. $R a_\times R^T = [Ra]_\times$

3. Lemm 4.

$$\Rightarrow \Phi_{v\theta}(t) = -{}_I\hat{R}_K \int_0^t \exp(s\omega_\times) a_\times \exp(s\omega_\times)^T ds$$

$$= -R\left[\int_0^t \exp(s\omega_\times) ds\, a\right]_\times$$

$$= -tR[J_L(t\omega)a]_\times$$

$\Phi_{v\omega}$ Proof:

$\because \omega_\times^3 = -\|\omega\|^2 \omega_\times$

$$\therefore\ I_3 - \frac{w_\times}{\|w\|^2}\left(\exp(w_\times) - I_3 - w_\times\right) = J_L(w).$$

We can also split Φ_{vw} into two parts:

$$\Phi_{vw}(t) = R\int_0^t s\exp(sw_\times)\,ds\, a_\times\left(I_3 + \frac{w_\times^2}{\|w\|^2}\right)$$
$$+ R\int_0^t \exp(sw_\times)\frac{a_\times w_\times}{\|w\|^2}\left(\exp(sw_\times)^T - I_3\right) ds$$

By integrating the terms of the Taylor series of $s\exp(sw_\times)$, and using $w_\times^3 = -\|w\|^2 w_\times$,

$$\int_0^t s\exp(sw_\times)\,ds = \sum_{n=0}^{\infty}\frac{t^{n+2}w_\times^n}{(n+2)\,n!}$$

$$= \frac{-1}{\|w\|^2}\sum_{n=0}^{\infty}\frac{t^{n+2}w_\times^{n+2}}{(n+2)\,n!}$$

$$= \frac{1}{\|w\|^2}\sum_{n=0}^{\infty}\left(\frac{t^{n+2}w_\times^{n+2}}{(n+2)!} - \frac{t^{n+2}w_\times^{n+2}}{(n+1)!}\right)$$

$$= \frac{\Delta(t)}{\|w\|^2}$$

where $\Delta t = \left(\exp(tw_\times)(I_3 - tw_\times) - I_3\right)$.

(See more details in a separate PDF in this blog).

The second integral can be computed using

$$\begin{cases} a_\times \omega_\times = \omega a^T - (a^T \omega) I_3 \\ \exp(s\omega_\times)\omega = \omega \\ \exp(s\omega_\times)\exp(s\omega_\times)^T = I_3 \\ \text{Lemma 4} \end{cases}$$

$$\therefore \Phi_{v\omega}(t) = R\Delta(t)\frac{a_\times}{\|\omega\|^2}\left(I_3 + \frac{\omega_\times^2}{\|\omega\|^2}\right)$$
$$+ tR\frac{\omega a^T}{\|\omega\|^2}(J_L(-t\omega) - I)$$
$$+ tR\frac{a^T\omega}{\|\omega\|^2}(J_L(t\omega) - I_3).$$

Φ_{va} Proof:

$$\begin{cases} \text{Solution to } {}_I\dot{\hat{R}} = {}_I\hat{R}\omega_\times \quad {}_I\hat{R}(0) = {}_I\hat{R}_k \\ \text{Lemma 4} \end{cases}$$

Finally, consider ${}_I\dot{\tilde{P}}(t) = {}_I\tilde{V}(t)$, which is an LTI system in ${}_I\tilde{P}(t)$, with solution

$$
{}_I\tilde{p}(t) = {}_I\tilde{p}(0) + \int_0^t {}_I\tilde{v}(s)\,ds
$$

$$
= [\,\Phi_{p\theta}(t) \;\; tI_3 \;\; I_3 \;\; \Phi_{pw}(t) \;\; \Phi_{pa}(t)\,]\; {}_I\tilde{x}(0).
$$

$$
\boxed{\begin{aligned}
\Phi_{p\theta}(t) &= \int_0^t \Phi_{v\theta}(s)\,ds = -t^2\hat{R}[H_L(tw)a]_\times \\
\Phi_{pw}(t) &= \int_0^t \Phi_{vw}(s)\,ds \\
\Phi_{pa}(t) &= \int_0^t \Phi_{va}(s)\,ds = -t^2\hat{R}H_L(tw)
\end{aligned}}
$$

$\Phi_{p\theta}, \Phi_{pa}$ follow from Lemma 4.

Φ_{pw} proof:

$$
\int_0^t \Delta(s)\,ds = \int_0^t \exp(sw_\times)\,ds - \int_0^t s\exp(sw_\times)\,ds\, w_\times - tI_3
$$

$$
= tJ_L(tw) - \frac{w_\times \Delta(t)}{\|w\|^2} - tI_3
$$

More details:

Quaternion kinematics for the error-state Kalman filter.

Appendix B.

Consider $\dot{x}(t) = A\cdot x(t)$.

assume the relation is linear and constant over the interval. $[t_n, t_n+\Delta t]$,

$$\Rightarrow \quad x_{n+1} = e^{A\Delta t} x_n = \Phi x_n.$$

Φ is transition matrix.

Taylor expansion of Φ:

$$\Phi = e^{A\cdot\Delta t} = I + A\Delta t + \frac{1}{2}A^2\Delta t^2 + \frac{1}{3!}A^3\Delta t^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \Delta t^k$$

B1. Integration of angular error

Consider the angular error dynamics without bias and noise:

$$\dot{\delta\theta} = -[w]_\times \delta\theta.$$

$$\Phi = e^{-[w]_\times \Delta t} = I - [w]_\times \Delta t + \frac{1}{2}[w]_\times^2 \Delta t^2 - \frac{1}{3!}[w]_\times^3 \Delta t^3 + \frac{1}{4!}[w]_\times^4 \Delta t^4 \ldots$$

define $w\Delta t \triangleq u\Delta\theta$.

$$\Phi = I - [u]_\times \Delta\theta + \frac{1}{2}[u]_\times^2 \Delta\theta^2 - \frac{1}{3!}[u]_\times^3 \Delta\theta^3 + \frac{1}{4!}[y]_\times^4 \Delta\theta^4 - \cdots$$
$$= I - [u]_\times \left(\Delta\theta - \frac{\Delta\theta^3}{3!} + \frac{\Delta\theta^5}{5!} - \cdots\right) + [u]_\times^2 \left(\frac{\Delta\theta^2}{2!} - \frac{\Delta\theta^4}{4!} + \frac{\Delta\theta^6}{6!} - \cdots\right)$$
$$= I - [u]_\times \sin\Delta\theta + [u]_\times^2 (1-\cos\Delta\theta).$$

B2. Simplified IMU example.

Error state dynamics:

$$\dot{\delta p} = \delta v$$

$$\dot{\delta v} = -R[a]_\times \delta\theta$$

$$\dot{\delta\theta} = -[\omega]_\times \delta\theta$$

The system is defined by the state vector and dynamic matrix:

$$x = \begin{bmatrix} \delta p \\ \delta v \\ \delta\theta \end{bmatrix}, \quad A = \begin{bmatrix} 0 & P_v & 0 \\ 0 & 0 & V_\theta \\ 0 & 0 & \Theta_\theta \end{bmatrix}$$

with

$$P_v = I$$

$$V_\theta = -R[a]_\times$$

$$\Theta_\theta = -[\omega]_\times$$

note here that R is constant.

In OrcVIO, R is $\hat{R}(t) = \hat{R}\exp(t\omega_\times)$

Its integration is $x_{n+1} = e^{(A\Delta t)} x_n = \Phi x_n$.

Let

$$\Phi = \begin{bmatrix} I & \Phi_{pv} & \Phi_{p\theta} \\ 0 & I & \Phi_{v\theta} \\ 0 & 0 & \Phi_{\theta\theta} \end{bmatrix}$$

Denote $R\{\phi\} \triangleq \mathrm{Exp}(\phi)$ $\quad \Phi_{\theta\theta} = R\{\omega \Delta t\}^T$.

$$\Phi_{v\theta} = -R[a]_\times \left(I\Delta t + \frac{[\omega]_\times}{\|\omega\|^2} \left(R\{\omega \Delta t\}^T - I + [\omega]_\times \Delta t \right) \right)$$