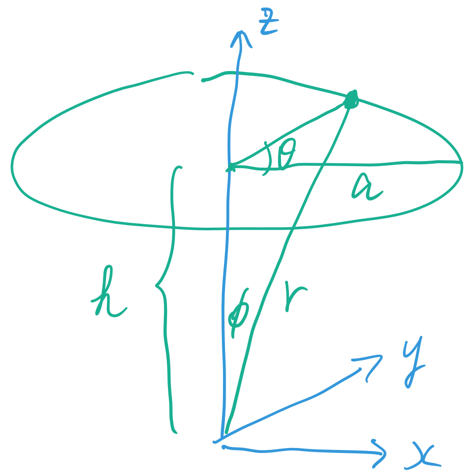
旋转运动学:



粒子坐标 $r=(a\cos\theta, a\sin\theta, h)^T$

对坐标求导:

$$\dot{r} = (-a\dot{\theta}\sin\theta, a\dot{\theta}\cos\theta, 0)^T \quad (1)$$

$$= \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a\cos\theta \\ a\sin\theta \\ h \end{bmatrix}$$

$$= \omega \times r$$

$\omega = \dot{\theta} z$, $|\dot{\theta}|$ 是角速度.

$$w = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

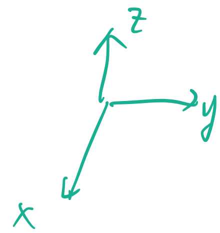
$|r|\sin\phi = a$

对(1)取模, $|\dot{r}| = |w|\,|r|\sin\phi = a|\dot{\theta}|$

线速度 = 半径 × 角速度.

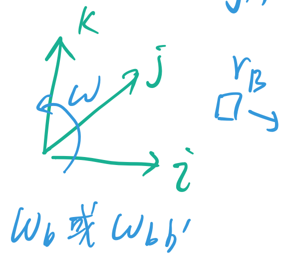
当坐标系由静止不动变为旋转时:

世界坐标系/惯性系：静止.



i,j,k 是世界系下的表示.

Body fram: 旋转的坐标系.



质量块在body下坐标:

$$r_B = (x_1, x_2, x_3)^T.$$

旋转到惯性系下:

$$r_I(t) = x_1(t)\, i + x_2(t)\, j + x_3(t)\, k = R_{IB}\, r_B$$

忽略两个坐标系间的平移.

简写为:

$$r_I = x_i e_i$$

对时间求导:

$$\dot{r}_I = R_{IB}\dot{r}_B + \dot{R}_{IB} r_B$$

$$= R_{IB}\dot{r}_B + [R_{IB}\omega_b]_\times r_I$$

(*) 推导见后面.

$$= R_{IB} v_B + \underbrace{\omega \times r_I}_{\text{extra item.}}$$

$$v_I \equiv R_{IB} v_B + \omega \times r_I \Rightarrow R_{IB} v_b \equiv v_I - \omega \times r_I \qquad (2).$$

$\omega = R_{IB}\omega_b$，表示body系角速度在I系表示.

(*) 推导:

$$\dot{R}_{IB} r_B = \lim_{\Delta t \to 0} \frac{\overbrace{R_{IB} \exp([\omega_{BB'} \Delta t]^\wedge)}^{R_{IB'}} r_B - R_{IB} r_B}{\Delta t}$$

$$R_{IB} \exp([\omega_{BB'} \Delta t])^\wedge) = R_{IB}(I + [\omega_b \Delta t]_\times) - R_{IB} r_B$$

$$= R_{IB}[\omega_b \Delta t]_\times$$

$\because R[a]_\times = [Ra]_\times R$

$\therefore R_{IB}[\omega_b \Delta t]_\times = [R_{IB}\omega_{BB'}]_\times R_{IB} r_B$

$$= \omega \times r_I$$

对速度求导:

$$v_I \equiv R_{IB} v_B + \omega \times r_I$$

$$\omega = R_{IB}\omega_b$$

$$\ddot{r} = (R_{IB}\dot{v}_B + \dot{R}_{IB} v_B) + (\omega \times \dot{r}_I + [\dot{R}_{IB}\omega_b + R_{IB}\dot{\omega}_b]_\times r_I)$$

把 $R_{IB} v_B$ 记作 v.

$$\dot{R}_{IB}\omega_b = \omega \times \omega = 0.$$

$$\dot{r}_I = v + \omega \times r_I$$

$$\therefore \ddot{r} = R_{IB}\dot{v}_B + \dot{R}_{IB} v_B + \omega \times (v + \omega \times r_I) + [R_{IB}\dot{\omega}_b]_\times r_I$$

$$= R_{IB} a_B + 2\omega \times v + \omega \times (\omega \times r_I) + \dot{\omega} \times r_I$$

$$R_{IB} v_B + \omega \times v = 2\omega \times v$$

把 $R_{IB} a_B$ 记作 a.

$$a = a_I - 2\omega \times v - \dot{\omega} \times r_I - \omega \times (\omega_I \times r_I)$$

科氏力　　欧拉力　　离心力.

IMU 测量模型.

加速度计测量值 a_m:

$$a_m = \frac{f}{m} = a - g.$$

a 物体在惯性系下的加速度.

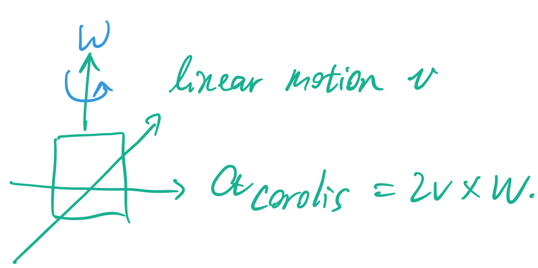
$$ma = f + mg. \to a = \frac{f}{m} + g.$$

东北天坐标系: $g = (0, 0, -9.81)^T$

IMU 静止: $a_m = 0 - g = -g$.

自由落体: $a = g$, $a_m = g - g = 0$.

MEMS 陀螺仪:

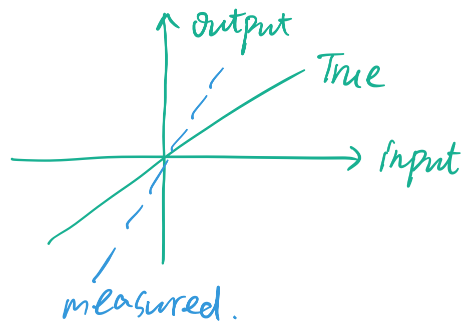


误差模型：

理论上没有外力作用时，IMU输出为0，但实际存在一个偏置b，

$$V_{err} = b_a t, \quad P_{err} = \frac{1}{2} b_a t^2.$$

Scale：实际数值和传感器输出之间的比值.



随机误差.

IMU数据在连续时间上受到一个均值为0，方差为σ^2，各时刻之间相互独立的高斯过程$n(t)$：

$$E[n(t)] = 0$$

$$E[n(t_1)n(t_2)] = \sigma^2 \delta(t_1 - t_2)$$

实际上，IMU传感器获取的数据为离散采样，离散和连续高斯白噪声的方差之间存在如下转换关系：

$$n_d[k] \triangleq n(t_0 + \Delta t) \simeq \frac{1}{\Delta t} \int^{t_0+\Delta t}$$

$$\Delta t \int_{t_0} n(\tau)\, dt$$

$$E(n_d[k]^2) = E\left(\frac{1}{\Delta t^2}\int_{t_0}^{t_0+\Delta t}\int_{t_0}^{t_0+\Delta t} n(\tau)\, n(t)\, d\tau\, dt\right)$$

$$= E\left(\frac{\sigma^2}{\Delta t^2}\int_{t_0}^{t_0+\Delta t}\int_{t_0}^{t_0+\Delta t} \delta(t-\tau)\, d\tau\, dt\right)$$

$$= E\left(\frac{\sigma^2}{\Delta t}\right)$$

$\therefore\ \sigma_d = \frac{\sigma}{\sqrt{\Delta t}}$ ， σ_d是离散采样后的标准差.

$$n_d[k] = \sigma_d\, w[k].$$

$$w[k] \sim N(0,1)$$

$$\sigma_d = \sigma \frac{1}{\sqrt{\Delta t}}.$$

bias 用 wiener process 建模，即随机游走：

$$\dot{b}(t) = n(t) = \sigma_b\, w(t).$$

$$w \sim N(0,1)$$

离散和连续之间的变换：

$$b_d[k] \triangleq b(t_0) + \int_{t_0}^{t_0+\Delta t} n(t)\, dt$$

$$E\left((b_d[k] - b_d[k-1])^2\right) =$$

$$= \int_{t_0}^{t_0+\Delta t}\int_{t_0}^{t_0+\Delta t}$$

$$E\left(\int_{t_0+\Delta t}\int_{t_0} n(t)\,n(\tau)\,d\tau\,dt\right) =$$

$$E\left(\sigma_b^2\int_{t_0}^{t_0+\Delta t}\int_{t_0}^{t_0+\Delta t}\delta(t-\tau)\,d\tau\,dt\right) =$$

$$E(\sigma_b^2\,\Delta t)$$

$$\therefore b_d[k] = b_d[k-1] + \sigma_{bd}\,w[k]$$

$$w[k] \sim N(0,1)$$

$$\sigma_{bd} = \sigma_b\sqrt{\Delta t}$$

加速度计数学模型.

导航系G为 ENU. $g^G=(0,0,-9.81)^T$.

$$a_M^B = R_{BG}(a^G - g^G) \qquad (16)$$

考虑高斯白噪声, bias, 及尺度因子:

$$a_m^B = S_a R_{BG}(a^G - g^G) + n_a + b_a. \qquad (17)$$

$n_a \sim N(0,\sigma_n^2)$ $\qquad$ $\dot{b}_a \sim N(0,\sigma_{ba}^2)$

陀螺仪误差:

$$\omega_m^B = S_g\omega^B + n_g + b_g.$$

运动模型离散时间处理: body frame

$$\boxed{\begin{aligned}\tilde{\omega}_b &= \omega^b + b^g + n^g \\ \tilde{a}^b &= q_{bw}(a^w + g^w) + b^a + n^a\end{aligned}}$$

测量值

w = world frame

PVQ对时间的导数:

$$\dot{p}_{wb_t} = v_t^w$$

$$\dot{v}_t^w = a_t^w$$

$$\dot{q}_{wb_t} = q_{wb_t} \otimes \begin{bmatrix} 0 \\ \frac{1}{2}\omega^{b_t} \end{bmatrix}$$

(需要归一化 q_{wb_t})

$$\dot{R}_{wb_t} = R_{wb_t} \omega^{b_t}_{\times}$$

根据导数关系,可从第 i 时刻的PVQ通过对IMU测量值积分,得到 j 时刻PVQ:

$$p_{wb_j} = p_{wb_i} + v_i^w \Delta t + \iint_{t\in[i,j]} (q_{wb_t} a^{b_t} - g^w)\, \delta t^2$$

$$v_j^w = v_i^w + \int_{t\in[i,j]} (q_{wb_t} a^{b_t} - g^w)\, \delta t$$

$$q_{wb_j} = \int_{t\in[i,j]} q_{wb_t} \otimes \begin{bmatrix} 0 \\ \frac{1}{2}\omega^{b_t} \end{bmatrix} \delta t.$$

使用欧拉法,即两个相邻时刻 k 到 $k+1$ 的位姿是用 k 时刻的测量值 a, ω 来计算

$$P_{wb_{k+1}} = P_{wb_k} + V_k^w \Delta t + \frac{1}{2} a \Delta t^2$$

$$V_{k+1}^w = V_k^w + a \Delta t$$

$$q_{wb_{k+1}} = q_{wb_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2} w \delta t \end{bmatrix}$$

$$\begin{cases} a = q_{wb_k}(a^{b_k} - b_k^a) - g^w \\ w = w^{b_k} - b_k^g \end{cases}$$

$$\dot{q} = q \otimes \begin{bmatrix} 0 \\ \frac{1}{2} w \end{bmatrix}$$

$$\begin{aligned} q_{k+1} &= q_k + \dot{q}_k \delta t \\ &= q_k + q_k \otimes \begin{bmatrix} 0 \\ \frac{1}{2} w \end{bmatrix} \delta t \\ &= q_k \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + q_k \otimes \begin{bmatrix} 0 \\ \frac{1}{2} w \end{bmatrix} \delta t \\ &= q_k \otimes \begin{bmatrix} 1 \\ \frac{1}{2} w \delta t \end{bmatrix} \end{aligned}$$

中值法：即两个时刻 k 到 k+1 位姿是用两时刻测量值 a, w 均值计算

$$\left.\begin{aligned} P_{wb_{k+1}} &= P_{wb_k} + V_k^w \Delta t + \frac{1}{2} a \Delta t^2 \\ V_{k+1}^w &= V_k^w + a \Delta t \\ q_{wb_{k+1}} &= q_{wb_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2} w \delta t \end{bmatrix} \end{aligned}\right\} \text{和欧拉法一样.}$$

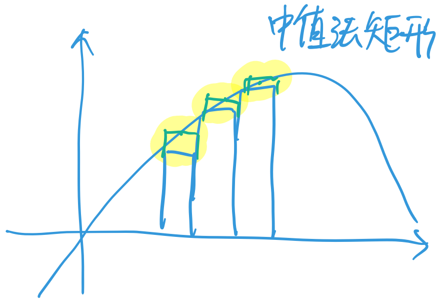
其中

和欧拉法不同.

$$a = \frac{1}{2}\left[q_{wb_k}(a^{b_k} - b_k^a) - g^w + q_{wb_{k+1}}(a^{b_{k+1}} - b_k^a) - g^w\right]$$

$$w = \frac{1}{2}\left[(w^{b_k} - b_k^g) + (w^{b_{k+1}} - b_k^g)\right]$$

中值法矩形比欧拉法大.



旋转积分:

$$q_{wb'} = q_{wb} \otimes \begin{bmatrix} 1 \\ \frac{1}{2} w \Delta t \end{bmatrix}$$

$$R_{wb'} = R_{wb} \exp(w \cdot \Delta t)$$

欧拉角: $\theta_{wb'} = \theta_{wb} + E_{wb} \cdot w \Delta t$.

$\theta = (\psi_{roll}, \theta_{pitch}, \phi_{yaw})^T$.

E_{wb} 将 IMU body系下角速度换成欧拉角速度.

欧拉角:

Step 1: 绕惯性系z轴, 得到新坐标 b'

$$x_b^1 = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} x_b^0 = R(\phi) x_b^0.$$

Step 2: 绕新坐标系 b' 的y轴旋转得到坐标系 b^2.

$$x_b^2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} x_b^1 = R(\theta)\, x_b^1 .$$

Step 3：绕 b^2 的x轴转得到 b^3.

$$x_b^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} x_b^2 = R(\psi)\, x_b^2$$

$$\therefore\ x_b = R(\psi)\, R(\theta)\, R(\phi)\, x = R(\phi, \theta, \psi)\, x .$$

body系下的x　　　　　　　　惯性系下的x.

角速度:

$$\omega = R(\psi)\, R(\theta) \begin{Bmatrix} 0 \\ 0 \\ \frac{d\phi}{dt} \end{Bmatrix} + R(\psi) \begin{Bmatrix} 0 \\ \frac{d\theta}{dt} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \frac{d\psi}{dt} \\ 0 \\ 0 \end{Bmatrix}$$

$$\omega^b = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\psi & \sin\psi\cos\theta \\ 0 & -\sin\psi & \cos\psi\cos\theta \end{bmatrix} \begin{bmatrix} \frac{d\psi}{dt} \\ \frac{d\theta}{dt} \\ \frac{d\phi}{dt} \end{bmatrix} \qquad (31)$$

(31) 取逆，得到 body rate to euler rate 的变换：

$$\frac{d\Theta}{dt} = \begin{bmatrix} 1 & \sin\psi\tan\theta & \cos\psi\tan\theta \\ 0 & \cos\psi & -\sin\psi \\ 0 & \frac{\sin\psi}{\cos\theta} & \frac{\cos\psi}{\cos\theta} \end{bmatrix} \vec{\omega}$$

twb