Bundle adjustment:

$$\underset{q,p,f}{\arg\min} \sum_{i=1}^{m} \sum_{j=1}^{n} \| \pi(q_{wc_i}, p_{wc_i}, f_j) - z_{f_j}^{c_i} \|_{\Sigma_{ij}}$$

特征点3D坐标.

最小二乘问题求解:

定义 $x^* \in R^n$, 使 $F(x)$ 局部最小.

$$F(x) = \frac{1}{2} \sum_{i=1}^{m} (f_i(x))^2$$

局部最小指 对任意, $\|x - x^*\| < \delta$ 有 $F(x^*) \le F(x)$

二阶泰勒展开:

$$F(x+\Delta x) = F(x) + J\Delta x + \frac{1}{2}\Delta x^T H \Delta x + O(\|\Delta x\|^3) \quad (2)$$

迭代法 1. 找到下降方向的单位向量 d.

2. 确定下降步长 α.

对 $F(x)$ 进行一阶泰勒展开:

$$F(x+\alpha d) \approx F(x) + \alpha J d$$

下降方向满足 $Jd < 0$.

通过 line search 找到下降步长:

$$\alpha^* = \underset{\alpha > 0}{\arg\min} \{F(x+\alpha d)\}$$

最速下降法寻找单位向量 d:

$$Jd = \|J\| \cos\theta, \ \theta \text{表示下降方向和梯度夹角.}$$

$$\theta = \pi \rightarrow d = -J^T$$

∴ 梯度的负方向为最速下降方向.

牛顿法: 在局部最优 x^* 附近, 如果 $x + \Delta x$ 最优, 则损失函数对 Δx 的导数为 0, 对 (2) 求一阶导:

$$\frac{\partial}{\partial \Delta x}\left(F(x) + J\Delta x + \frac{1}{2}\Delta x^T H \Delta x\right)$$

$$= J^T + H\Delta x = 0 \rightarrow \Delta x = -H^{-1}J^T.$$

二阶导矩阵计算复杂.

阻尼牛顿法: 二阶泰勒展开:

$$F(x + \Delta x) \approx L(\Delta x) \equiv F(x) + J\Delta x + \frac{1}{2}\Delta x^T H \Delta x$$

$$\Delta x \equiv \underset{\Delta x}{\arg\min}\left\{L(\Delta x) + \frac{1}{2}\mu \Delta x^T \Delta x\right\}.$$

$\mu \geq 0$ 为阻尼因子.

$\frac{1}{2}\mu \Delta x^T \Delta x = \frac{1}{2}\mu \|\Delta x\|^2$ 是惩罚项. 限制 Δx 不要走太远

求一阶导并令其为 0:

$$L'(\Delta x) + \mu \Delta x = 0$$

$$\Rightarrow (H+\mu I)\Delta x = -J^T$$

非线性最小二乘:

$$f(x) = \begin{bmatrix} f_1(x) \\ \cdots \\ f_m(x) \end{bmatrix}$$

$$\therefore f^T(x) f(x) = \sum_{i=1}^{m} (f_i(x))^2$$

令 $J_i(x) = \frac{\partial f_i(x)}{\partial x}$, 则有

$$\frac{\partial f(x)}{\partial x} = J = \begin{bmatrix} J_1(x) \\ \cdots \\ J_m(x) \end{bmatrix}$$

残差 $f(x)$ 为非线性, 一阶泰勒展开:

$$f(x+\Delta x) \approx l(\Delta x) \equiv f(x) + J\Delta x$$

J是残差函数 f 的雅可比矩阵.

代入损失函数:

$$F(x+\Delta x) \approx L(\Delta x) \equiv \frac{1}{2} l(\Delta x)^T l(\Delta x)$$

$$= \frac{1}{2} f^T f + \Delta x^T J^T f + \frac{1}{2} \Delta x^T J^T J \Delta x$$

$$= F(x) + \Delta x^T J^T f + \frac{1}{2} \Delta x^T J^T J \Delta x \qquad (7)$$

$\therefore$ 损失函数近似成二次函数, 并且如果雅可比满秩, J^TJ 正定,

损失函数有最小值.　　$Jx=0$ 等价于 $x=0$.

令(7)导数为 0:

$$J^T f + J^T J \Delta x = 0$$

$$\therefore (J^T J) \Delta x_{gn} = -J^T f.$$

用 $J^T J$ 近似 H: $H \Delta x_{gn} = b$.

normal equation.

$\because F(x+\Delta x) \approx F(x) + \Delta x^T J^T f$, $\therefore F'(x) = (J^T f)^T$

$F(x+\Delta x) \approx F(x) + \Delta x^T J^T f + \frac{1}{2} \Delta x^T J^T J \Delta x$. $\therefore F''(x) = J^T J$.

LM: 对高斯牛顿法加入阻尼, 限制 Δx 不要跑太远.

$$(J^T J + \mu I) \Delta x_{lm} = -J^T f, \quad \mu \geq 0.$$

$\mu > 0$ 保证 $(J^T J + \mu I)$ 正定, 迭代朝下降方向进行.

$\mu >> 0$, $\Delta x_{lm} = -\frac{1}{\mu} J^T f$. 接近最速下降.

$\mu << 0$, $\Delta x_{lm} \approx \Delta x_{gn}$, 接近高斯牛顿.

阻尼因子更新策略通过比例因子确定:

$$\rho = \frac{F(x) - F(x + \Delta x_{lm})}{L(0) - L(\Delta x_{lm})}$$

$$b = -J^T f$$

$$L(0) - L(\Delta x_{lm}) = -\Delta x_{lm}^T J^T f - \frac{1}{2}\Delta x_{lm}^T J^T J \Delta x_{lm}$$

$$= -\frac{1}{2}\Delta x_{lm}^T \left(-2b + (J^T J + \mu I - \mu I)\Delta x_{lm}\right)$$

$$= \frac{1}{2}\Delta x_{lm}^T (\mu \Delta x_{lm} + b) > 0.$$

$$\therefore \rho = \frac{F(x) - F(x+\Delta x_{lm})}{\frac{1}{2}\Delta x_{lm}^T (\mu \Delta x_{lm} + b)}$$

$\rho < 0$, 则 $F(x)\uparrow$, 应 $\mu\uparrow \rightarrow \Delta x\downarrow$. 增大阻尼减小步长.

$\rho >> 0$, 减小 μ, 让LM接近GN, 使收敛加快.

$\rho > 0$ 但 ρ 较小, 则增大 μ, 减小步长.

鲁棒核函数.

$$\min_x \frac{1}{2}\sum_k \rho\left(\|f_k(x)\|^2\right).$$

误差平方项记为 $s_k = \|f_k(x)\|^2$, 则鲁棒核函数二阶泰勒展开:

$$\frac{1}{2}\rho(s) = \frac{1}{2}\left(\text{const} + \rho' \Delta s + \frac{1}{2}\rho'' \Delta^2 s\right)$$

对变量 Δx 求导, 令其为0:

$$\sum_k J_k^T (\rho' I + 2\rho'' f_k f_k^T) J_k \Delta x = -\sum_k \rho' f_k^T J_k.$$

$$\sum_k J_k^T W J_k \Delta x = -\sum_k \rho' f_k^T J_k.$$

GN: $J^T J \Delta x_{gn} = -J^T f$