基于滑窗的VIO:

$$\min_{\mathcal{X}} \rho\left(\|r_p - J_p \mathcal{X}\|^2_{\Sigma_p}\right) + \sum_{i\in B} \rho\left(\|r_b(z_{b_i b_{i+1}}, \mathcal{X})\|^2_{\Sigma_{b_i b_{i+1}}}\right)$$

prior IMU error

$$+ \sum_{(i,j)\in F} \rho\left(\|r_f(z^{c_i}_{f_j}, \mathcal{X})\|^2_{\Sigma^{c_i}_{f_j}}\right)$$

image error.

在i时刻,滑窗内状态量为:

$$\mathcal{X} = [x_n, x_{n+1}, \cdots, x_{n+N}, \lambda_m, \lambda_{m+1} \ldots \lambda_{m+M}]$$

$$x_i = [P_{wb_i}, q_{wb_i}, v_i^w, b_a^{b_i}, b_g^{b_i}], \quad i \in [n, n+N]$$

N: 关键帧. M: 路标. λ: 逆深度.

归一化平面下误差:

$$r_c = \begin{bmatrix} \frac{x}{z} - u \\ \frac{y}{z} - v \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$\lambda = \frac{1}{z}$ 是逆深度.

特征点在i帧初始化，在j帧被观测：

$$\begin{bmatrix} x_{c_j} \\ y_{c_j} \\ z_{c_j} \\ 1 \end{bmatrix} = T_{bc}^{-1} T_{wb_j}^{-1} T_{wb_i} T_{bc} \begin{bmatrix} \frac{1}{\lambda} u_{c_i} \\ \frac{1}{\lambda} v_{c_i} \\ \frac{1}{\lambda} \\ 1 \end{bmatrix}$$

residual：

$$r_c = \begin{bmatrix} \frac{x_{c_j}}{z_{c_j}} - u_{c_j} \\ \frac{y_{c_j}}{z_{c_j}} - v_{c_j} \end{bmatrix}$$

IMU误差

$$\tilde{w}^b = w^b + b^g + n^g$$

$$\tilde{a}^b = q_{bw}(a^w + g^w) + b^a + n^a$$

对时间的导数

$$\dot{p}_{wb_t} = v_t^w$$

$$\dot{v}_t^w = a_t^w$$

$$\dot{q}_{wb_t} = q_{wb_t} \otimes \begin{bmatrix} 0 \\ \frac{1}{2} w^{b_t} \end{bmatrix}$$

从i时刻对测量积分，得到j时刻PVQ：

$$p_{wb_j} = p_{wb_i} + v_i^w \Delta t + \iint_{t\in[i,j]} (q_{wb_t} a^{b_t} - g^w)\, \delta t^2$$

$$v_j^w = v_i^w + \int_{t\in[i,j]} (q_{wb_t} a^{b_t} - g^w)\, \delta t$$

$$q_{wb_j} = \int_{t\in[i,j]} q_{wb_t} \otimes \begin{bmatrix} 0 \\ \frac{1}{2}\omega^{b_t} \end{bmatrix} \delta t$$

每次更新状态量都需要重新积分，计算量大、

IMU 预积分：

一个简单的公式转换，就可以转为预积分：

$$q_{wb_t} = q_{wb_i} \otimes q_{b_i b_t}$$

把世界系下变量取出，PVQ 积分公式中积分项变成相对于第 i 时刻的姿态，

而不是世界系下姿态：

$$p_{wb_j} = p_{wb_i} + v_i^w \Delta t - \frac{1}{2} g^w \Delta t^2 + q_{wb_i} \iint_{t\in[i,j]} (q_{b_i b_t} a^{b_t})\, \delta t^2$$

$$v_j^w = v_i^w - g^w \Delta t + q_{wb_i} \int_{t\in[i,j]} (q_{b_i b_t} a^{b_t})\, \delta t$$

$$q_{wb_j} = q_{wb_i} \int_{t\in[i,j]} q_{b_i b_t} \otimes \begin{bmatrix} 0 \\ \frac{1}{2}\omega^{b_t} \end{bmatrix} \delta t$$

预积分量仅和 IMU 测量值有关：

$$\alpha_{b_i b_j} = \iint_{t\in[i,j]} (q_{b_i b_t} a^{b_t})\, \delta t^2$$

$$\beta_{b_ib_j} = \int_{t\in[i,j]} (q_{b_ib_t} a^{b_t}) \delta t$$

$$q_{b_ib_j} = \int_{t\in[i,j]} q_{b_ib_t} \otimes \begin{bmatrix} 0 \\ \frac{1}{2}\omega^{b_t} \end{bmatrix} \delta t$$

重新整理 PVQ 积分公式:

$$\begin{bmatrix} P_{wb_j} \\ v_j^w \\ q_{wb_j} \\ b_j^a \\ b_j^g \end{bmatrix} = \begin{bmatrix} P_{wb_i} + v_i^w \Delta t - \frac{1}{2} g^w \Delta t^2 + q_{wb_i} \alpha_{b_ib_j} \\ v_i^w - g^w \Delta t + q_{wb_i} \beta_{b_ib_j} \\ q_{wb_i} q_{b_ib_j} \\ b_i^a \\ b_i^g \end{bmatrix}$$

IMU 预积分误差: 一段时间内 IMU 构建的预积分对两时刻间状态量进行约束:

$$\begin{bmatrix} r_p \\ r_q \\ r_v \\ r_{ba} \\ r_{bg} \end{bmatrix} = \begin{bmatrix} q_{b_iw}(P_{wb_j} - P_{wb_i} - v_i^w \Delta t + \frac{1}{2} g^w \Delta t^2) - \alpha_{b_ib_j} \\ 2[q_{b_jb_i} \otimes (q_{b_iw} \otimes q_{wb_j})]_{xyz} \leftarrow \text{只取虚部} \\ q_{b_iw}(v_j^w - v_i^w + g^w \Delta t) - \beta_{b_ib_j} \\ b_j^a - b_i^a \\ b_j^g - b_i^g \end{bmatrix}$$

$$\left.\begin{matrix} b_j^a - b_i^a \\ b_j^g - b_i^g \end{matrix}\right\} \text{短时间内相等.}$$

离散形式，用中值法，即k到k+1的位姿是用测量值的均值计算.

$$\omega = \frac{1}{2}\left((\omega^{b_k} - b_k^g) + (\omega^{b_{k+1}} - b_k^g)\right)$$

$$q_{b_i b_{k+1}} = q_{b_i b_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega\delta t \end{bmatrix}$$

$$a = \frac{1}{2}\left(q_{b_i b_k}(a^{b_k} - b_k^a) + q_{b_i b_{k+1}}(a^{b_{k+1}} - b_k^a)\right)$$

$$\alpha_{b_i b_{k+1}} = \alpha_{b_i b_k} + \beta_{b_i b_k}\delta t + \frac{1}{2}a\delta t^2$$

$$\beta_{b_i b_{k+1}} = \beta_{b_i b_k} + a\delta t$$

$$b_{k+1}^a = b_k^a + n_{b_k}^a \delta t$$

$$b_{k+1}^g = b_k^g + n_{b_k}^g \delta t$$

Covariance propagation.

已知 $y = Ax$，$x \in N(0, \Sigma_x)$，则有 $\Sigma_y = A\Sigma_x A^T$.

$$\begin{aligned} \Sigma_y &= E\left((Ax)(Ax)^T\right) \\ &= E(Axx^TA^T) \\ &= A\Sigma_x A^T \end{aligned}$$

∴要推导预积分的协方差，要知道 imu 噪声和预积分之间的线性传递.

假设相邻时刻线性传递方程为:

$$\eta_{ik} = F_{k-1}\eta_{ik-1} + G_{k-1} n_{k-1}.$$

$$\eta_{ik} = [\delta\theta_{ik}, \delta v_{ik}, \delta p_{ik}].$$

$$n_k = [n_k^g, n_k^a]$$

$$\Rightarrow \Sigma_{ik} = F_{k-1}\Sigma_{ik-1}F_{k-1}^T + G_{k-1}\Sigma_n G_{k-1}^T$$

Σ_n 是测量噪声的协方差，从 i 时刻开始递推，$\Sigma_{ii} = 0$.

基于一阶泰勒展开的误差递推：

$$x = \hat{x} + \delta x, \quad \hat{x} \text{ 为真值},\ \delta x \text{ 为误差}.$$

$x_k = f(x_{k-1}, u_{k-1})$ 的递推如下：

$$\delta x_k = F\delta x_{k-1} + G n_{k-1}.$$

F：x_k 对 x_{k-1} 的雅可比.

G：x_k 对 u_{k-1} 的雅可比.

证明：

$$x_k = f(x_{k-1}, u_{k-1})$$

输入量受高斯噪声影响.

$$\hat{x}_k + \delta x_k = f(\hat{x}_{k-1} + \delta x_{k-1}, \hat{u}_{k-1} + n_{k-1})$$

$$\hat{x}_k + \delta x_k = f(\hat{x}_{k-1}, \hat{u}_{k-1}) + F\delta x_{k-1} + G n_{k-1}.$$

负值抵消.

基于随时间变化的递推:

$$\dot{\delta x} = A\delta x + Bn.$$

$$\delta x_k = \delta x_{k-1} + \dot{\delta x}_{k-1}\Delta t$$

$$\Rightarrow \delta x_k = (I + A\Delta t)\delta x_{k-1} + B\Delta t\, n_{k-1}$$

$$\Rightarrow F = I + A\Delta t,\quad G = B\Delta t.$$

之所以用时间变化的递推,是因为已知速度和状态量之间的关系:

$$\dot{v} = Ra^b + g.$$

$$\dot{p} = v$$

就可推出速度误差和状态误差之间关系:

$$\dot{v} - \dot{\delta v} = R(I + [\delta\theta]_\times)(a^b + \delta a^b) + g + \delta g$$

$$\dot{\delta v} = R\delta a^b + R[\delta\theta]_\times(a^b + \delta a^b) + \delta g.$$

$$\dot{\delta v} = R\delta a^b - R[a^b]_\times\delta\theta + \delta g.$$

以下采用更通用的泰勒展开推导. 回顾误差递推公式:

$$\omega = \frac{1}{2}\left((\omega^{b_k} + n_k^g - b_k^g) + (\omega^{b_{k+1}} + n_{k+1}^g - b_k^g)\right)$$

$$q_{b_ib_{k+1}} = q_{b_ib_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega \delta t \end{bmatrix}$$

$$a = \frac{1}{2}\left(q_{b_ib_k}(a^{b_k} + n_k^a - b_k^a) + q_{b_ib_{k+1}}(a^{b_{k+1}} + n_{k+1}^g - b_k^a)\right)$$

$$\alpha_{b_ib_{k+1}} = \alpha_{b_ib_k} + \beta_{b_ib_k}\delta t + \frac{1}{2} a \delta t^2$$

$$\beta_{b_ib_{k+1}} = \beta_{b_ib_k} + a \delta t$$

$$b_{k+1}^a = b_k^a + n_{b_k}^a \delta t$$

$$b_{k+1}^g = b_k^g + n b_k^g \delta t$$

我们希望推导出如下关系:

$$\begin{bmatrix} \delta\alpha_{b_{k+1}b'_{k+1}} \\ \delta\theta_{b_{k+1}b'_{k+1}} \\ \delta\beta_{b_{k+1}b'_{k+1}} \\ \delta b_{k+1}^a \\ \delta b_{k+1}^g \end{bmatrix} = F \begin{bmatrix} \delta\alpha_{b_kb'_k} \\ \delta\theta_{b_kb'_k} \\ \delta\beta_{b_kb'_k} \\ \delta b_k^a \\ \delta b_k^g \end{bmatrix} + G \begin{bmatrix} n_k^a \\ n_k^g \\ n_{k+1}^a \\ n_{b_k}^a \\ n_{b_k}^g \end{bmatrix}$$

β对各状态的雅可比，即F第三行.

$$\beta_{b_ib_{k+1}} = \beta_{b_ib_k} + a\delta t$$

加速度的中值积分

$$= \beta_{b_ib_k} + \frac{1}{2}\left(q_{b_ib_k}(a^{b_k} - b_k^a) + q_{b_ib_{k+1}}(a^{b_{k+1}} - b_k^a)\right)\delta t. \quad (47)$$

$$f_{33} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \delta \beta_{b_k b_k'}} = I_{3\times 3}$$

对于 f_{32}，(47) 写为：

$$\beta_{b_i b_{k+1}} = \beta_{b_i b_k} + \frac{1}{2}\left(q_{b_i b_k}(a^{b_k} - b_k^a) + \underbrace{q_{b_i b_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega \delta t \end{bmatrix}}_{q_{b_i b_{k+1}}} (a^{b_{k+1}} - b_k^a)\right)\delta t$$

$$\frac{\partial \beta_{b_i b_{k+1}}}{\partial \delta \theta_{b_k b_k'}} = \frac{\partial a \delta t}{\partial \delta \theta_{b_k b_k'}}$$

扰动

$$a\delta t = \frac{1}{2} q_{b_i b_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta \theta_{b_k b_k'} \end{bmatrix} (a^{b_k} - b_k^a)\delta t$$

$$+ \frac{1}{2} q_{b_i b_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta \theta_{b_k b_k'} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega \delta t \end{bmatrix} (a^{b_{k+1}} - b_k^a)\delta t$$

$$= \frac{1}{2} R_{b_i b_k} \exp([\delta \theta_{b_k b_k'}]_\times)(a^{b_k} - b_k^a)\delta t \quad ①$$

$$+ \frac{1}{2} R_{b_i b_k} \exp([\delta \theta_{b_k b_k'}]_\times) \exp([\omega \delta t]_\times)(a^{b_{k+1}} - b_k^a)\delta t. \quad ②$$

① 对应雅可比为：

$$\frac{\partial R_{b_i b_k} \exp([\delta \theta_{b_k b_k'}]_\times)(a^{b_k} - b_k^a)\delta t}{\partial \delta \theta_{b_k b_k'}}$$

$$= \frac{\partial R_{b_i b_k} (I + [\delta \theta_{b_k b_k'}]_\times)(a^{b_k} - b_k^a)\delta t}{\partial \delta \theta_{\ldots}}$$

$\partial \delta\theta_{b_k b_k'}$

$$= \frac{\partial -R_{b_i b_k}[(a^{b_k}-b_k^a)\delta t]_\times \delta\theta_{b_k b_k'}}{\partial\delta\theta_{b_k b_k'}}$$

$$= -R_{b_i b_k}[(a^{b_k}-b_k^a)\delta t]_\times$$

② 对应的雅可比：

$$\frac{\partial R_{b_i b_k}\exp([\delta\theta_{b_k b_k'}]_\times)\exp([w\delta t]_\times)(a^{b_{k+1}}-b_k^a)\delta t}{\partial\delta\theta_{b_k b_k'}}$$

$$= \frac{\partial R_{b_i b_k}(I+[\delta\theta_{b_k b_k'}]_\times)\exp([w\delta t]_\times)(a^{b_{k+1}}-b_k^a))\delta t}{\partial\delta\theta_{b_k b_k'}}$$

$$\approx -R_{b_i b_{k+1}}[(a^{b_k}-b_k^a)\delta t]_\times(I-[w\delta t]_\times)$$

残差雅可比的推导.

视觉残差.

$$r_c = \begin{bmatrix} \frac{x_{c_j}}{z_{c_j}} - u_{c_j} \\ \frac{y_{c_j}}{z_{c_j}} - v_{c_j} \end{bmatrix}$$

$$\begin{bmatrix} x_{c_j} \\ y_{c_j} \end{bmatrix} \quad \quad \begin{bmatrix} \frac{1}{\lambda}u_{c_i} \\ \frac{1}{\lambda}v_{c_i} \end{bmatrix}$$

$$
\begin{bmatrix} \cdot_{c_j} \\ z_{c_j} \\ 1 \end{bmatrix} = T_{bc}^{-1} T_{wb_j}^{-1} T_{wb_i} T_{bc} \begin{bmatrix} \cdot \\ \frac{1}{\lambda} \\ 1 \end{bmatrix}
$$

$$
\Rightarrow f_{c_j} = \begin{bmatrix} x_{c_j} \\ y_{c_j} \\ z_{c_j} \end{bmatrix} = R_{bc}^T R_{wb_j}^T R_{wb_i} R_{bc} \frac{1}{\lambda} \begin{bmatrix} u_{c_i} \\ v_{c_i} \\ 1 \end{bmatrix}
$$

$$
+ R_{bc}^T \left(R_{wb_j}^T \left((R_{wb_i} p_{bc} + p_{wb_i}) - p_{wb_j} \right) - p_{bc} \right)
$$

定义:

$$
f_{b_i} = R_{bc} f_{c_i} + p_{bc}
$$

$$
f_w = R_{wb_i} f_{b_i} + p_{wb_i}
$$

$$
f_{b_j} = R_{wb_j}^T (f_{c_i} - p_{bc})
$$

Jacobian为视觉误差对两个时刻的状态量,外参,以及逆深度求导:

$$
J = \begin{bmatrix} \dfrac{\partial r_c}{\partial \begin{bmatrix} \delta p_{b_i b_i'} \\ \delta \theta_{b_i b_i'} \end{bmatrix}} & \dfrac{\partial r_c}{\partial \begin{bmatrix} \delta p_{b_j b_j'} \\ \delta \theta_{b_j b_j'} \end{bmatrix}} & \dfrac{\partial r_c}{\partial \begin{bmatrix} \delta p_{cc'} \\ \delta \theta_{cc'} \end{bmatrix}} & \dfrac{\partial r_c}{\partial \delta \lambda} \end{bmatrix}
$$

根据链式法则:

① $$
\frac{\partial r_c}{\partial f_{c_i}} = \begin{bmatrix} \frac{1}{z_{c_j}} & 0 & -\frac{x_{c_j}}{z_{c_j}^2} \\ \cdots \end{bmatrix}
$$

$$\begin{bmatrix} 0 & \frac{1}{z_{c_j}} & -\frac{y_{c_j}}{z_{c_j}^2} \end{bmatrix}$$

② 1. 对 i 时刻状态求导.

$$\frac{\partial f_{c_j}}{\partial \delta p_{b_i b_i'}} = R_{bc}^T R_{wb_j}^T$$

$$f_{c_j} = R_{bc}^T R_{wb_j}^T R_{wb_i} R_{bc} f_{c_i} + R_{bc}^T \left(R_{wb_j}^T \left(\left(R_{wb_i} p_{bc} + p_{wb_i} \right) - p_{wb_j} \right) - p_{bc} \right)$$

$$\begin{aligned}
\Rightarrow f_{c_j} &= R_{bc}^T R_{wb_j}^T R_{wb_i} R_{bc} f_{c_i} + R_{bc}^T R_{wb_j}^T R_{wb_i} p_{bc} + \cdots \\
&= R_{bc}^T R_{wb_j}^T R_{wb_i} \left(R_{bc} f_{c_i} + p_{bc} \right) + \cdots \\
&= R_{bc}^T R_{wb_j}^T R_{wb_i} f_{b_i} + \cdots
\end{aligned}$$

$$\Rightarrow \frac{\partial f_{c_j}}{\partial \delta \theta_{b_i b_i'}} = \frac{\partial R_{bc}^T R_{wb_j}^T R_{wb_i} \left(I + [\delta \theta_{b_i b_i'}]_\times \right) f_{b_i}}{\partial \delta \theta_{b_i b_i'}}$$

$$= -R_{bc}^T R_{wb_j}^T R_{wb_i} [f_{b_i}]_\times$$

② 2. 对 j 时刻状态求导.

$$\frac{\partial f_{c_j}}{\partial \delta p_{b_j b_j'}} = -R_{bc}^T R_{wb_j}^T$$

$$f_{c_j} = R_{bc}^T R_{wb_j}^T (f_w - p_{wb_j}) + \cdots$$

$$\frac{\partial f_{c_j}}{\partial \delta \theta_{b_i b_i'}} = \frac{\partial R_{bc}^T \left(I - [\delta \theta_{b_j b_j'}]_\times \right) R_{wb_j}^T (f_w - p_{wb_j})}{\partial \delta \theta_{b_j b_j'}}$$

$$
= \frac{\partial R_{bc}^{T}\left(I - [\delta\theta_{b_jb_j'}]_{\times}\right) f_{b_j}}{\partial \delta\theta_{b_jb_j'}}
$$

$$
= R_{bc}^{T}[f_{b_j}]_{\times}
$$

IMU误差对于优化变量的Jacobian.

对于预积分在i时刻的bias，用一阶泰勒展开

$$
\alpha_{b_ib_j} = \alpha_{b_ib_j} + J^{\alpha}_{b_i^a}\delta b_i^a + J^{\alpha}_{b_i^g}\delta b_i^g
$$

$$
\beta_{b_ib_j} = \beta_{b_ib_j} + J^{\beta}_{b_i^a}\delta b_i^a + J^{\beta}_{b_i^g}\delta b_i^g
$$

$$
q_{b_ib_j} = q_{b_ib_j} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}J^{q}_{b_i^g}\delta b_i^g \end{bmatrix}
$$